Homework #1
Due Su 9/10

Note:
OW  Oppenheim and Wilsky
SSS  Schaum’s Signals and Systems
SPR  Schaum’s Probability, Random Variables, and Random Processes

Be sure to show all your work for credit.

1. (SSS 1.51)

Solution

(a) \( x(t) = \cos(2t + \pi/4) \) This is a sinusoid so periodic. The period can be found as
\[
\omega_0 = 2 \quad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi.
\]

(b) \( x(t) = \cos^2(t) \) The sinusoid \( \cos(t) \) is periodic with period \( T = 2\pi \). The square inverts negative portion of the cosine and effectively halves the period
\[
T = \pi.
\]

(c) \( x(t) = \cos(2\pi t)u(t) \) This cannot be periodic because of the \( u(t) \) term making the signal zero for \( t < 0 \).

(d) \( x(t) = e^{j\pi t} \) This is a complex periodic exponential by definition
\[
\omega_0 = \pi \quad T = \frac{2\pi}{\omega_0} = 2.
\]

(e) \( x[n] = e^{j(n/4 - \pi)} \) This is a complex exponential so can be periodic. However, in this case, the \( \frac{2\pi}{\omega_0} \) is not rational meaning the period will not be an integer. Hence, it is not periodic.
\[
N = \frac{2\pi}{\omega_0} = \frac{2\pi}{1/4} = 8\pi.
\]

(f) \( x[n] = \cos(\frac{\pi n^2}{8}) \)
\[
x[n] = \cos\left(\frac{\pi n^2}{8}\right) = \frac{1}{2} e^{j\frac{\pi n^2}{8}} + \frac{1}{2} e^{-j\frac{\pi n^2}{8}}.
\]
This problem is a little more complicated and not something that would be expected for a exam. Using Matlab it is easy to see that
\[
N = 8.
\]

The outline of a more formal approach is outlined below:
\[
x[n] = \cos\left(\frac{\pi n^2}{8}\right) = x[n + N] = \cos\left(\frac{\pi (n + N)^2}{8}\right)
\]

Multiply out the right side and use Euler’s
\[
\frac{1}{2} \left[ e^{j\frac{\pi n^2}{8}} + e^{-j\frac{\pi n^2}{8}} \right] = \frac{1}{2} \left[ e^{j\frac{\pi n^2}{8}} e^{j\frac{2\pi N}{8}} e^{j\pi N^2} + e^{-j\frac{\pi n^2}{8}} e^{-j\frac{2\pi N}{8}} e^{-j\pi N^2} \right]
\]
Due to symmetry, only the $+j$ terms need to be considered

$$e^{j\frac{2nN}{8}} e^{j\frac{N^2}{8}} = 1 = e^{j2\pi k}$$

$$e^{j\frac{2nN}{8} + \frac{N^2}{8}} = e^{j2\pi k}$$

$$\Rightarrow 2nN + N^2 = 16k \quad \forall n$$

Ensuring that both $2nN$ and $N^2$ are multiples of 16, the smallest value of $N$ is 8.

(g) $x[n] = \cos(n/2) \cos(\pi n/4)$ This is not periodic because $\cos(n/2)$ does not have a rational period.

(h) $x[n] = \cos(\pi n/4) + \sin(\pi n/8) - 2 \cos(\pi n/2)$ The first cos is $N_1 = 8$, the second sin has $N_2 = 16$, and the third cos has $N_3 = 4$. The period of $x[n]$ is

$$N_0 = \text{lcm}(N_1, N_2, N_3) = 16.$$

2. (SSS 1.56)

Solution

$$y(t) = T\{x(t)\} = \frac{1}{T} \int_{t-T/2}^{t+T/2} x(\tau) d\tau$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

where

$$h(t) = \begin{cases} 1 & -T/2 \leq t \leq T/2 \\ 0 & \text{else} \end{cases}.$$ 

Since this can be written in a convolution form, it is Linear and Time-Invariant. The system is not causal, because $h(t) \neq 0$ for $n < 0$.

3. (SSS 1.57)

Solution

$$y(t) = T\{x(t)\} = \sum_{k=-\infty}^{\infty} x(t) \delta(t - kT)$$

$$= \sum_{k} x(kT) \delta(t - kT).$$

[Linear] Checking for linearity

$$T\{ax_1(t) + bx_2(t)\} = \sum_{k} (ax_1(t) + bx_2(t)) \delta(t - kT)$$

$$= a \sum_{k} x_1(kT) \delta(t - kT) + b \sum_{k} x_2(kT) \delta(t - kT)$$

$$= aT\{x_1(t)\} + bT\{x_2(t)\}.$$
[Not TI] Checking time-invariance

\[ y_1(t) = y(t - t_0) = \sum_k x(t - t_0)\delta(t - t_0 - kT) \]
\[ = \sum_k x(kT)\delta(t - t_0 - kT) \]
\[ y_2(t) = T\{x(t - t_0)\} = \sum_k x(t - t_0)\delta(t - kT) \]
\[ = \sum_k x(kT - t_0)\delta(t - kT) \]

not TI since \( y_1(t) \neq y_2(t) \).

4. (SSS 1.58)

Solution

\[ y[n] = T\{x[n]\} = x^2[n] \]

[Not Linear] Checking linearity

\[ y[n] = T\{x[n]\} = (ax_1[n] + bx_2[n])^2 = a^2x_1^2[n] + b^2x_2^2[n] + abx_1[n]x_2[n] \]
\[ \neq ax_1[n] + bx_2[n] \]

[Time Invariant] Checking time-invariance

\[ y_1[n] = y[n - n_0] = x^2[n - n_0] \]
\[ y_2[n] = T\{x[n - n_0]\} = (x[n - n_0])^2 \]

5. (SSS 2.46 (b), (c))

Solution

(b) Choosing to flip \( h(t) \) for convolution there are five cases.

case 1: \( t < 0 \)
\[ y(t) = 0 \]
case 2: \( t - 2T > T \Rightarrow t > 3T \)
\[ y(t) = 0 \]
case 3: \( 0 \leq t \leq T \)
\[ y(t) = \int_0^t \tau d\tau = \left[ \frac{1}{2} \tau^2 \right]_0^t = \frac{1}{2} t^2 \]
case 4: \( t > T \) and \( t - 2T \leq 0 \) \( \Rightarrow T < t \leq 2T \)

\[
y(t) = \int_0^T \tau d\tau = \frac{1}{2} T^2
\]

case 5: \( 2T < t \leq 3T \)

\[
y(t) = \int_{t-2T}^T \tau d\tau = \frac{1}{2} [t^2]_t-2T
\]

\[
= \frac{1}{2} (T^2 - (t - 2T)^2)
\]

\[
= \frac{1}{2} (T^2 - (t^2 - 4Tt + 4T^2))
\]

\[
= \frac{1}{2} (-3T^2 - t^2 + 4Tt)
\]

(c) Choosing to flip \( x(t) = u(t-1) \) for convolution there are two cases.

case 1: \( t - 1 < 0 \)

\[
y(t) = 0
\]

case 2: \( t \geq 1 \)

\[
y(t) = \int_0^{t-1} e^{-3\tau} d\tau = -\frac{1}{3} [e^{-3\tau}]_0^{t-1}
\]

\[
= \frac{1}{3} [1 - e^{-3(t-1)}].
\]

The final result incorporating both cases is simply

\[
y(t) = \frac{1}{3} \left[ 1 - e^{-3(t-1)} \right] u(t-1).
\]

6. (SSS 2.47 (a), (c))

Solution

(a) Choosing to flip \( h[n] \) for convolution there are two cases.

case 1: \( n > 0 \)

\[
y[n] = \sum_{k=n}^{\infty} 2^{n-k} = 2^n \sum_{k=n}^{\infty} \left(\frac{1}{2}\right)^k
\]

\[
= 2^n \frac{\left(\frac{1}{2}\right)^n}{1 - 1/2} = 2
\]
case 2: \( n \leq 0 \)

\[
y[n] = \sum_{k=0}^{\infty} 2^{n-k} \\
= 2^n \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k = 2^n \times 2 \\
= 2^{n+1}
\]

(c)

\[
y[n] = x[n] * h[n] = x[n] - \frac{1}{2} x[n-1] \\
= \left( \frac{1}{2} \right)^n u[n] - \frac{1}{2} \left( \frac{1}{2} \right)^{n-1} u[n-1] \\
= \left( \frac{1}{2} \right)^n u[n] - \left( \frac{1}{2} \right)^n u[n-1]
\]

Since only the first sample at \( n = 0 \) survives

\[
y[n] = \delta[n].
\]

7. (SSS 2.62)

Solution

There are many ways to approach this problem but the easiest is probably using the z-transform which results in the following equations (where \( w[n] \) is the node above the delay):

\[
Y(z) = 2W(z) + z^{-1}W(z) = W(z)[2 + z^{-1}] \\
W(z) = X(z) + \frac{1}{2} z^{-1}W(z)
\]

This relationship results in an equation for \( X(z) \)

\[
X(z) = W(z)[1 - \frac{1}{2} z^{-1}]
\]

Plugging together

\[
Y(z) = \frac{2 + z^{-1}}{1 - \frac{1}{2} z^{-1}} X(z)
\]

The final difference equation can be found by cross-multiplying and taking the inverse z-transform

\[
y[n] - \frac{1}{2} y[n-1] = 2x[n] + x[n-1].
\]

8. (OW 1.36)

(a) Utilizing the periodicity constraint
\[ x[n + N] = x[n] \]
\[ = x[(n + N)T] = e^{j\omega_0(n+N)T} = e^{j\omega_0nT}e^{j\omega_0NT} \]
\[ = x[n] = e^{j\omega_0nT} \]
\[ \Rightarrow e^{j\omega_0NT} = 1 = e^{j2\pi k} \]
\[ \Rightarrow \omega_0NT = 2\pi k \quad \forall k \in \mathbb{Z} \]
\[ \frac{2\pi}{T_0}NT = 2\pi k \]
\[ \frac{T}{T_0} = \frac{k}{N} \Rightarrow \text{rational number because both } k, N \text{ are integers} \]

(b) \[ x[n] = e^{j\omega_0nT} = e^{j2\pi nT/T_0} = e^{j2\pi n(p/q)} \]
Using the result from problem 1.35,

Fundamental Period
\[ N_0 = \frac{q}{\gcd(p, q)} \]

Fundamental Frequency
\[ \Omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{q} \gcd(p, q) = \frac{\omega_0T}{p} \gcd(p, q) \]

(c) For one period we want
\[ \omega_0N_0T = 2\pi \]
\[ \frac{2\pi}{T_0}N_0T = 2\pi \]
\[ \Rightarrow \frac{T}{T_0} = 1 \]
\[ \frac{p}{q} \frac{q}{\gcd(p, q)} = 1 \]
\[ \Rightarrow \frac{p}{q} \frac{q}{\gcd(p, q)} = 1 \quad \text{period} \]

9. (OW 1.37)

Solution

(a)
\[ \phi_{xy}(t) = \int_{-\infty}^{\infty} x(t + \tau)y(\tau)d\tau \]
\[ \phi_{yx}(t) = \int_{-\infty}^{\infty} x(t')y(-t + t')d\tau \]
\[ = \phi_{yx}(-t) \]

(b) From part a) we know that this is an even function so the odd part is zero.
\[ \phi_{xx}(t) = \phi_{xx}(-t). \]
(c) $y(t) = x(t + T)$

$$\phi_{xy}(t) = \int_{-\infty}^{\infty} x(t + \tau)x(\tau + T)d\tau$$
$$= \int_{-\infty}^{\infty} x(t + T + t')x(t')dt'$$
$$= \phi_{xx}(t - T)$$

$$\phi_{yy}(t) = \int_{-\infty}^{\infty} y(t + \tau)x(\tau + T)d\tau$$
$$= \int_{-\infty}^{\infty} x(t + T + \tau)x(\tau + T)d\tau$$
$$= \int_{-\infty}^{\infty} x(t + t')x(t')dt'$$

$$\phi_{yy}(t) = \phi_{xx}(t).$$

10. (OW 2.50)

Solution

(a)

$$y_3(t) = ay_1(t) + by_2(t) \iff x_3(t) = B\{ay_1(t) + by_2(t)\} = ax_1(t) + bx_2(t).$$

(b)

$$x_1(t - \tau) \iff y_1(t - \tau) \iff x_1(t - \tau).$$