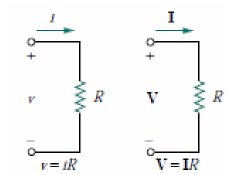
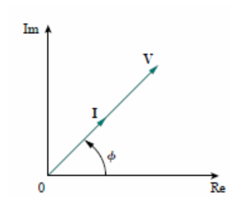
Phasor Relationships for Resistors

- Each circuit element has a relationship between its current and voltage.
- These can be mapped into phasor relationships very simply for resistors capacitors and inductor.
- For the resistor, the voltage and current are related via Ohm's law.
- As such, the voltage and current are in phase with each other.





Phasor Relationships for Resistors

If the current through a resistor *R* is:

$$i = I_m \cos(\omega t + \phi),$$

the voltage across it is given by Ohm's law as:

$$v = iR = RI_m \cos(\omega t + \phi)$$

The phasor form of this voltage is:

$$\mathbf{V} = RI_m \angle \phi$$

But the phasor representation of the current is: $\mathbf{I} = I_m \angle \phi$.

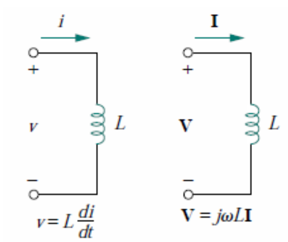
Hence, **V** = **RI**.

Phasor Relationships for Inductors

- Inductors on the other hand have a phase shift between the voltage and current.
- In this case, the voltage leads the current by 90°.
- Or one says the current lags the voltage, which is the standard convention.
- This is represented on the phasor diagram by a positive phase angle between the voltage and current.

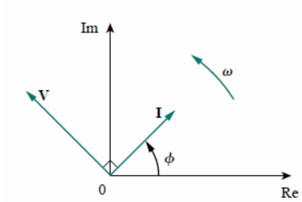
For the inductor L, assume the current through it is:

$i = \quad I_m \cos(\omega t + \phi).$



The voltage across the inductor is:

$v = \quad v = L\dfrac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$



(But, - sin A = cos (A+90°))

$v = \omega L I_m \cos(\omega t + \phi + 90^\circ)$

which transforms to the phasor: $\mathbf{V} = \omega L I_m e^{j(\phi+90^\circ)} = \omega L I_m e^{j\phi} e^{j90^\circ} = \omega L I_m \angle\phi\, e^{j90^\circ}$

But $I_m \angle\phi = \mathbf{I}$, and $e^{j90^\circ} = j$ **Therefore, V = $j\omega L$I**

Phasor Relationships for Capacitors

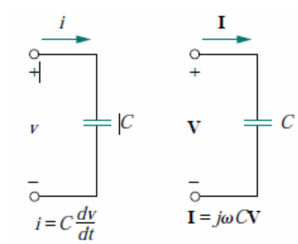
- Capacitors have the opposite phase relationship as compared to inductors.
- In their case, the current leads the voltage.
- In a phasor diagram, this corresponds to a negative phase angle between the voltage and current.

Phasor Relationships for Capacitors

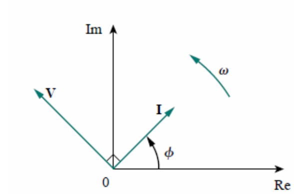
For the capacitor *C*, assume the voltage across it is
v = $V_m \cos(\omega t + \phi)$.
The current through the capacitor is:

$$i = C\frac{dv}{dt}$$

$$\mathbf{I} = j\omega C\mathbf{V} \quad \Longrightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$



The current leads the voltage by 90°



Voltage current relationships

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Impedance and Admittance

- It is possible to expand Ohm's law to capacitors and inductors.
- In time domain, this would be tricky as the ratios of voltage and current and always changing.
- But in frequency domain it is straightforward
- The impedance of a circuit element is the ratio of the phasor voltage to the phasor current.

$$Z = \frac{V}{I} \quad \text{or} \quad V = ZI$$

- Admittance is simply the inverse of impedance.

The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).

The impedance represents the opposition which the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

As a complex quantity, the impedance may be expressed in rectangular form as $\mathbf{Z} = R + jX$

$$\mathbf{Z} = R + jX$$

The reactance X may be positive or negative. We say that the impedance is inductive when X is positive or capacitive when X is negative. Thus, impedance $\mathbf{Z} = R + jX$ is said to be *inductive* or lagging since current lags voltage, while impedance $\mathbf{Z} = R - jX$ is capacitive or leading because current leads voltage. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in polar form as $\mathbf{Z} = |\mathbf{Z}| \underline{/\theta}$

$$\mathbf{Z} = R + jX = |\mathbf{Z}| \underline{/\theta}$$

where

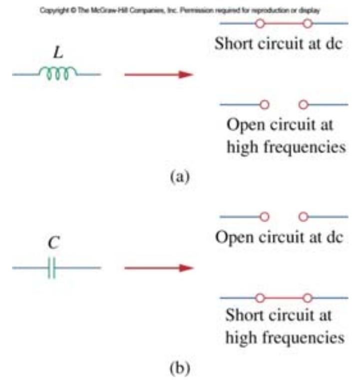
$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R}$$

and

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta$$

Impedance and Admittance

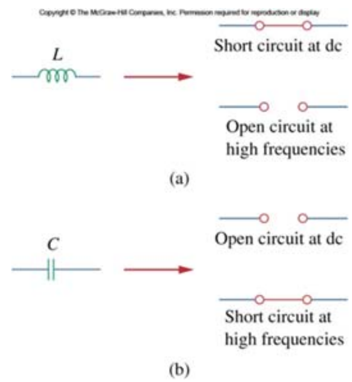
- It is important to realize that in frequency domain, the values obtained for impedance are only valid at that frequency.
- Changing to a new frequency will require recalculating the values.
- The impedance of capacitors and inductors are shown here:



Impedance and Admittance

- As a complex quantity, the impedance may be expressed in rectangular form.
- The separation of the real and imaginary components is useful.
- The real part is the resistance.
- The imaginary component is called the reactance, *X*.
- When it is positive, we say the impedance is inductive, and capacitive when it is negative.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$



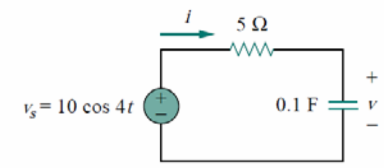
Impedance and Admittance

- Admittance, being the reciprocal of the impedance, is also a complex number.
- It is measured in units of Siemens
- The real part of the admittance is called the conductance, *G*
- The imaginary part is called the susceptance, *B*
- These are all expressed in Siemens or (mhos)
- The impedance and admittance components can be related to each other:

$$G = \frac{R}{R^2 + X^2} \qquad B = -\frac{X}{R^2 + X^2}$$

Impedance and Admittance

Find *v(t)* and *i(t)* in the circuit shown in Fig.



From the voltage source $10\cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\underline{/0^\circ}\ \text{V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\underline{/0^\circ}}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$
$$= 1.6 + j0.8 = 1.789\underline{/26.57^\circ}\ \text{A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\underline{/26.57^\circ}}{j4 \times 0.1}$$
$$= \frac{1.789\underline{/26.57^\circ}}{0.4\underline{/90^\circ}} = 4.47\underline{/-63.43^\circ}\ \text{V}$$

Converting **I** and **V** to the time domain, we get:

$$i(t) = 1.789\cos(4t + 26.57^\circ)\ \text{A}$$
$$v(t) = 4.47\cos(4t - 63.43^\circ)\ \text{V}$$

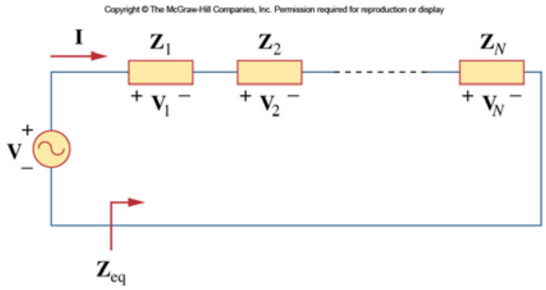
Notice that $i(t)$ leads $v(t)$ by 90° as expected.

Kirchoff’s Laws in Frequency Domain

- A powerful aspect of phasors is that Kirchoff’s laws apply to them as well.
- This means that a circuit transformed to frequency domain can be evaluated by the same methodology developed for KVL and KCL.
- One consequence is that there will likely be complex values.

Impedance Combinations

- Once in frequency domain, the impedance elements are generalized.
- Combinations will follow the rules for resistors:



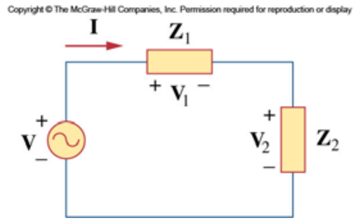
Impedance Combinations

- Series combinations will result in a sum of the impedance elements:

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots + Z_N$$

- Here then two elements in series can act like a voltage divider

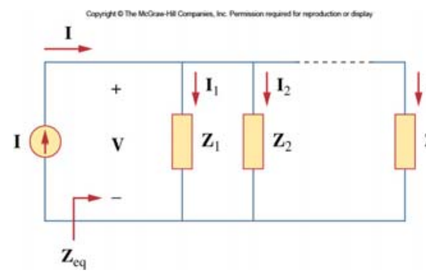
$$V_1 = \frac{Z_1}{Z_1 + Z_2} V \quad V_2 = \frac{Z_2}{Z_1 + Z_2} V$$



Parallel Combination

- Likewise, elements combined in parallel will combine in the same fashion as resistors in parallel:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots + \frac{1}{Z_N}$$



Admittance

- Expressed as admittance, though, they are again a sum:

$$Y_{eq} = Y_1 + Y_2 + Y_3 + \cdots + Y_N$$

- Once again, these elements can act as a current divider:

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I \qquad I_2 = \frac{Z_1}{Z_1 + Z_2} I$$

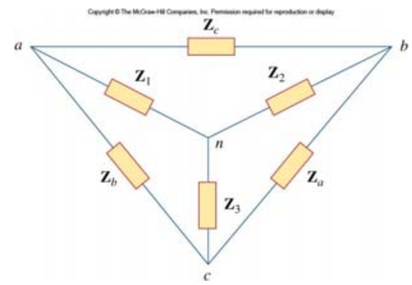
Impedance Combinations

- The Delta-Wye transformation is:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c} \qquad Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c} \qquad Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} \qquad Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$



Homework: Chapter 9

30, 34, 42, 52, 60, 64.

Due: Sept. 15.