EE482/682: DSP APPLICATIONS

CH4 IIR FILTER DESIGN

INTRODUCTION

CHAPTER 4.1

IIR DESIGN

- Reuse well studied analog filter design techniques (books and tables for design)
- Need to map between analog design and a digital design
  - Mapping between s-plane and z-plane

ANALOG BASICS

- Laplace transform
  - $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$
- Complex s-plane
  - $s = \sigma + j\Omega$
    - Complex number with σ and Ω real
  - $j\Omega$ – imaginary axis
- Fourier transform for $\sigma = 0$
  - When region of convergence contains the $j\Omega$ axis
- Convolution relationship
  - $y(t) = x(t) * h(t) \rightarrow Y(s) = X(s)H(s)$
  - $H(s) = \frac{Y(s)}{X(s)} = \int_{-\infty}^{\infty} h(t)e^{-st}\, dt$
- Stability constraint requires poles to be in the left half s-plane

MAPPING PROPERTIES

- Z-transform from Laplace by change of variable
  - $z = e^{sT} = e^{\sigma T} e^{j\Omega T} = |z| e^{j\omega}$
    - $|z| = e^{\sigma T},\ \omega = \Omega T$
- This mapping is not unique
  - $-\pi/T < \Omega < \pi/T$ → unit circle
  - 2π multiples as well
  - Left half s-plane mapped inside unit circle
  - Right half s-plane mapped outside unit circle

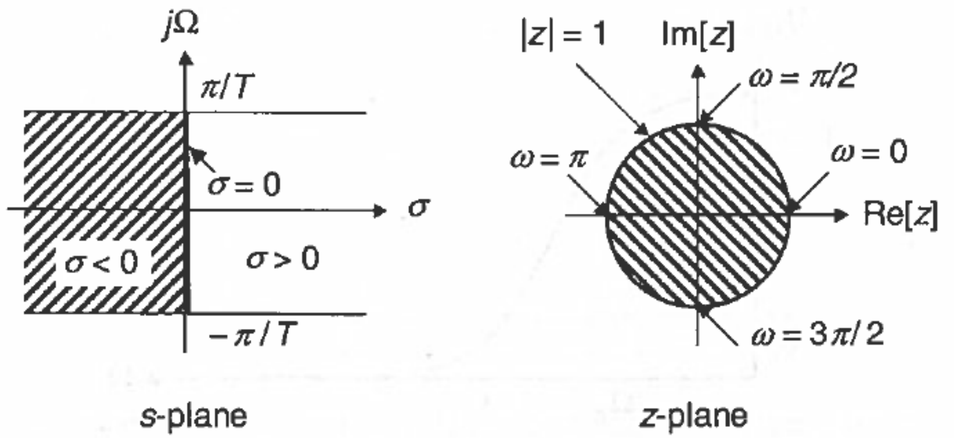


Figure 4.1 Mapping properties between the s-plane and the z-plane

FILTER CHARACTERISTICS

- Designed to meet a given/desired magnitude response
- Trade-off between:
  - Phase response
  - Roll-off rate – how steep is the transition between pass and stopband (transition width)

BUTTERWORTH FILTER

- All-pole approximation to ideal filter

- $|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_p)^{2L}}$
  - $|H(0)| = 1$
  - $|H(\Omega_p)| = 1/\sqrt{2}$
    - -3 dB @ Ω_p
- Has flat magnitude response in pass and stopband (no ripple)
- Slow monotonic transition band
  - Generally needs larger L

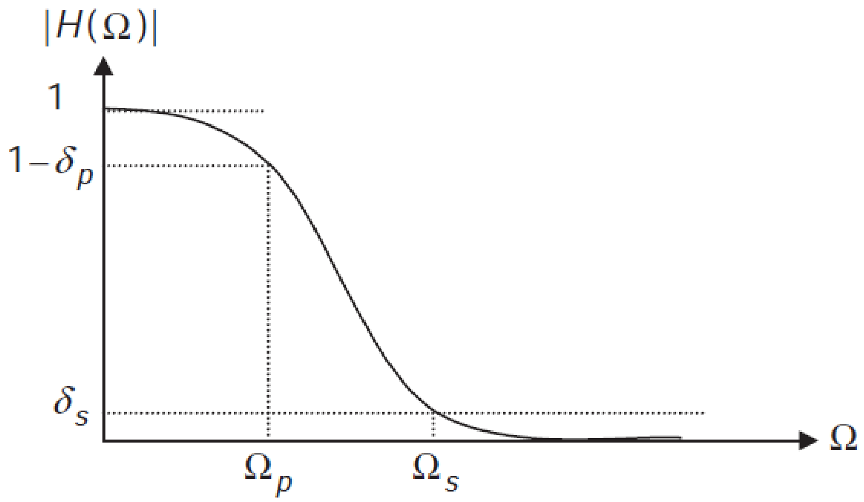


Figure 4.2 Magnitude response of Butterworth lowpass filter

CHEBYSHEV FILTER

- Steeper roll-off at cutoff frequency than Butterworth
  - Allows certain number of ripples in either passband or stopband
- Type I – equiripple in passband, monotonic in stopband
  - All-pole filter
- Type II – equiripple in stopband, monotinic in passband
  - Poles and zeros
- Generally better magnitude response than Butterworth but at cost of poorer phase response

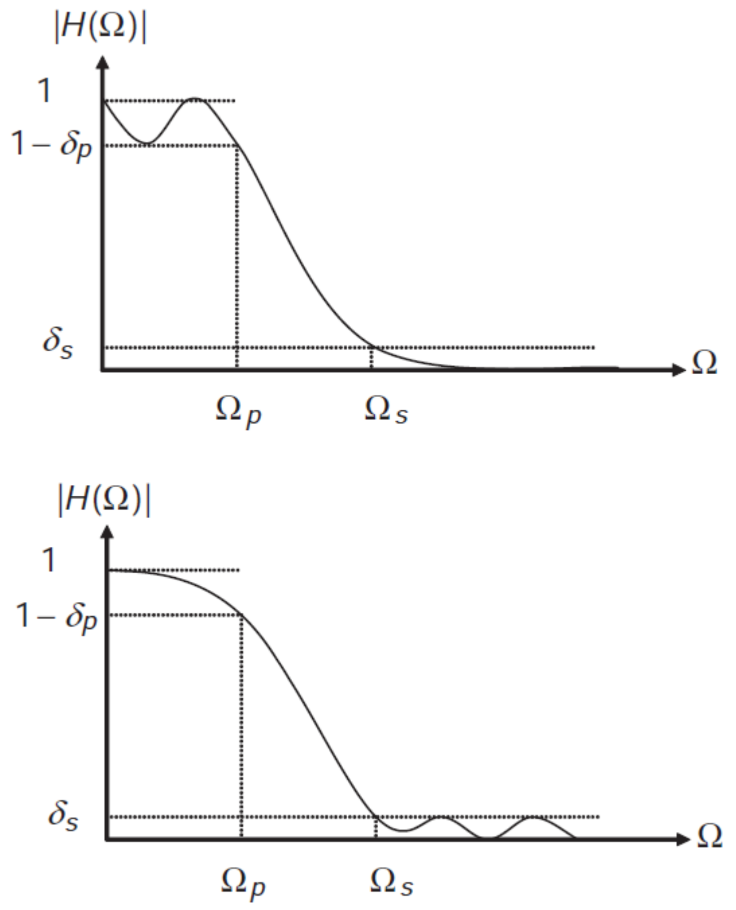


Figure 4.3 Magnitude responses of Chebyshev type I (top) and type II lowpass filters

ELLIPTIC FILTER

- Sharpest passband to stopband transition
- Equiripple in both pass and stopbands
- Phase response is highly non-linear in passband
  - Should only be used in situations where phase is not important to design

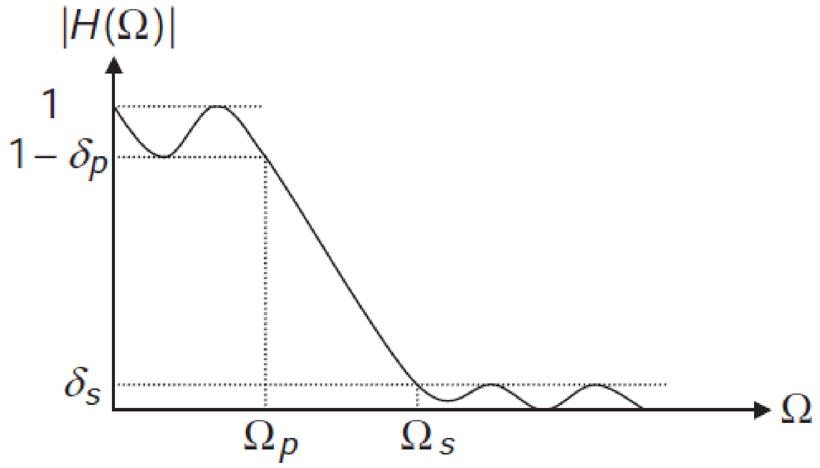


Figure 4.4 Magnitude response of elliptic lowpass filter

FREQUENCY TRANSFORMS

- Design lowpass filter and transform from LP to another type (HP, BP, BS)
- Define mapping
- $H(z) = H_{lp}(Z)|_{Z^{-1}=G(z^{-1})}$
  - Replace Z^{-1} in LP filter with $G(z^{-1})$
- θ – frequency in LP
- ω – frequency in new transformed filter

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency

DESIGN OF IIR FILTERS

CHAPTER4.2

IIR FILTER DESIGN

- IIR transfer function

$$H(z) = \frac{\sum_{l=0}^{L-1} b_l z^{-l}}{1 + \sum_{l=0}^{M} a_l z^{-l}}$$

- Need to find coefficients a_l, b_l
  - Impulse invariance – sample impulse response
    - Have to deal with aliasing
  - Bilinear transform
    - Match magnitude response
    - "Warp" frequencies to prevent aliasing

BILINEAR TRANSFORM DESIGN

- Convert digital filter into an "equivalent" analog filter
  - Use bilinear "warping"
- Design analog filter using IIR design techniques
- Map analog filter into digital
  - Use bilinear transform

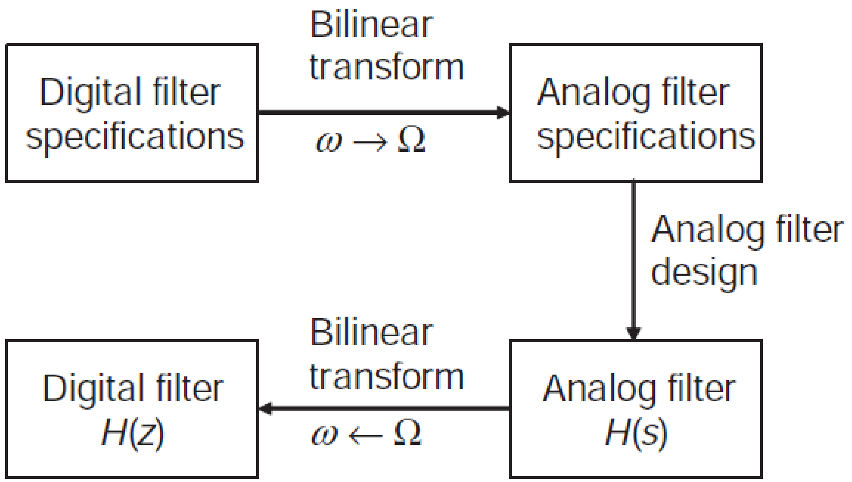


Figure 4.5 Digital IIR filter design using the bilinear transform

BILINEAR TRANSFORMATION

- Mapping from s-plane to z-plane
- $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$
- Frequency mapping
  - $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$
  - $\omega = 2\arctan\left(\frac{\Omega T}{2}\right)$
- Entire $j\omega$-axis is squished into $[-\pi/T, \pi/T]$ to prevent aliasing
  - Unique mapping
  - Highly non-linear which requires "pre-warp" in design

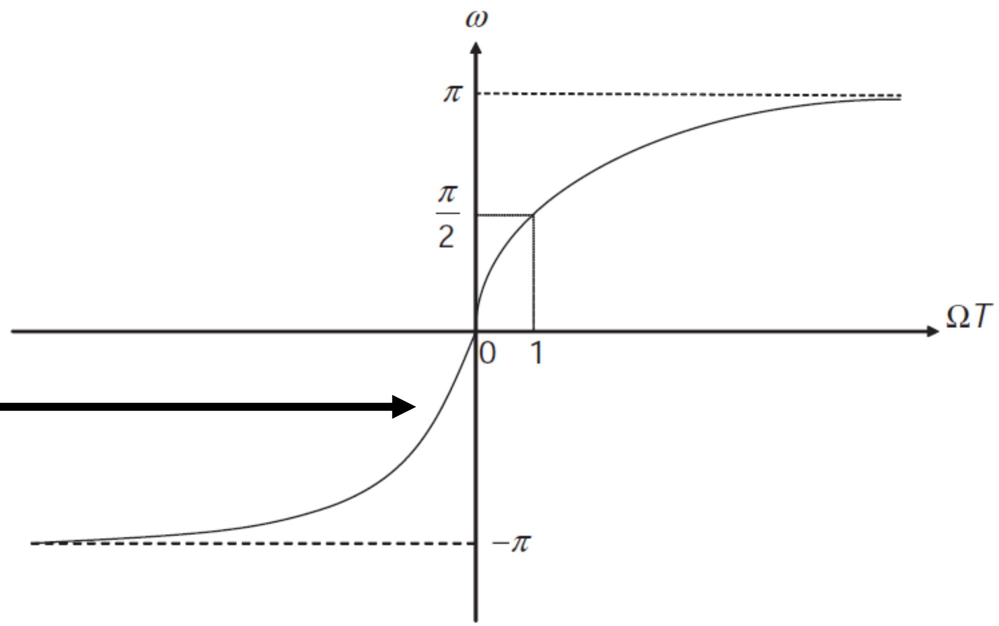
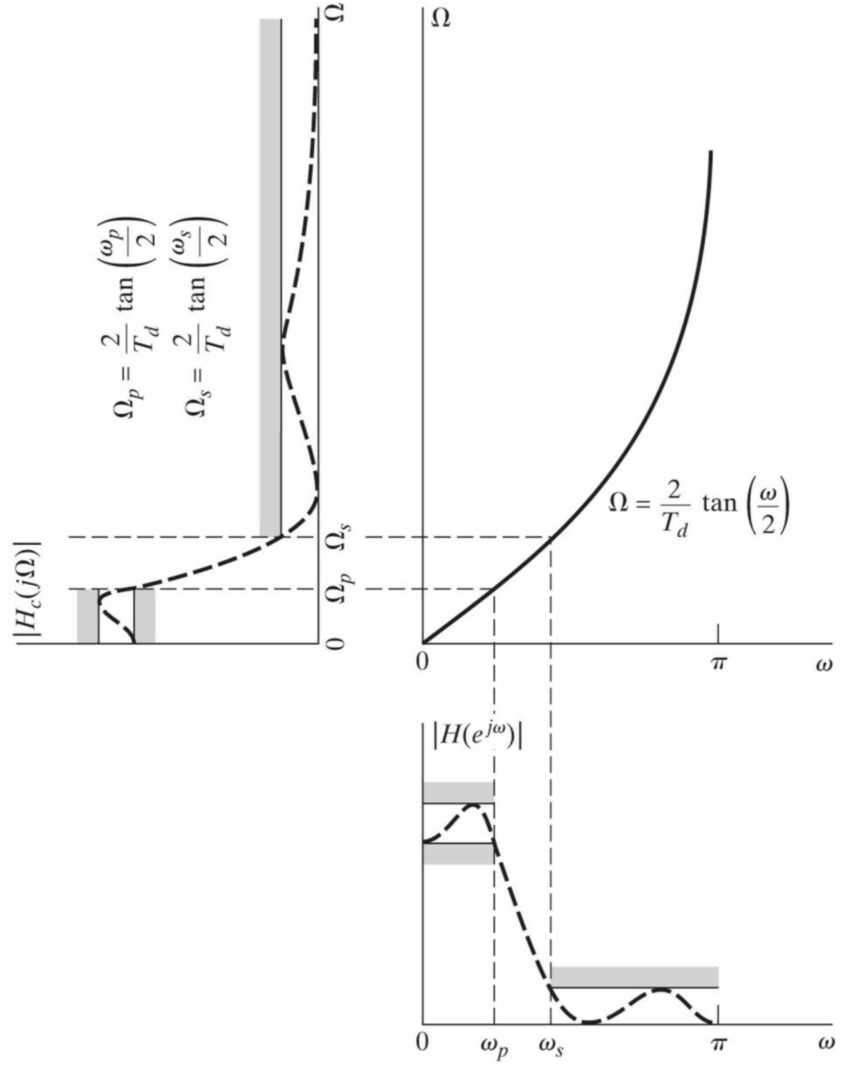


Figure 4.6 Frequency warping of bilinear transform defined by (4.27)

BILINEAR DESIGN STEPS

- Convert digital filter into an "equivalent" analog filter
  - Pre-warp using: $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$
- Design analog filter using IIR design techniques
  - Butterworth, Chebyshev, Elliptical
- Map analog filter into digital
  - $H(z) = H(s)\big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$



BILINEAR DESIGN EXAMPLE

- Example 4.2
- Design filter using bilinear transform
  - $H(s) = 1/(s+1)$
  - Bandwith 1000 Hz
  - $f_s = 8000$ Hz
- DT parameters
  - $\omega_c = 2\pi(1000/8000) = 0.25\pi$

1. Pre-warp: $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$
   - $\Omega_c = \frac{2}{T}\tan(0.125\pi) = \frac{0.8284}{T}$
2. Scale frequency (normalize scale)
   - $\hat{H}(s) = H\left(\frac{s}{\Omega_c}\right) = \frac{0.8284}{sT+0.8284}$
3. Bilinear transform:
   - $H(z) = H(s)\big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$
   - $H(z) = \frac{0.2929\left(1+z^{-1}\right)}{1-0.4141z^{-1}}$

REALIZATION OF IIR FILTERS

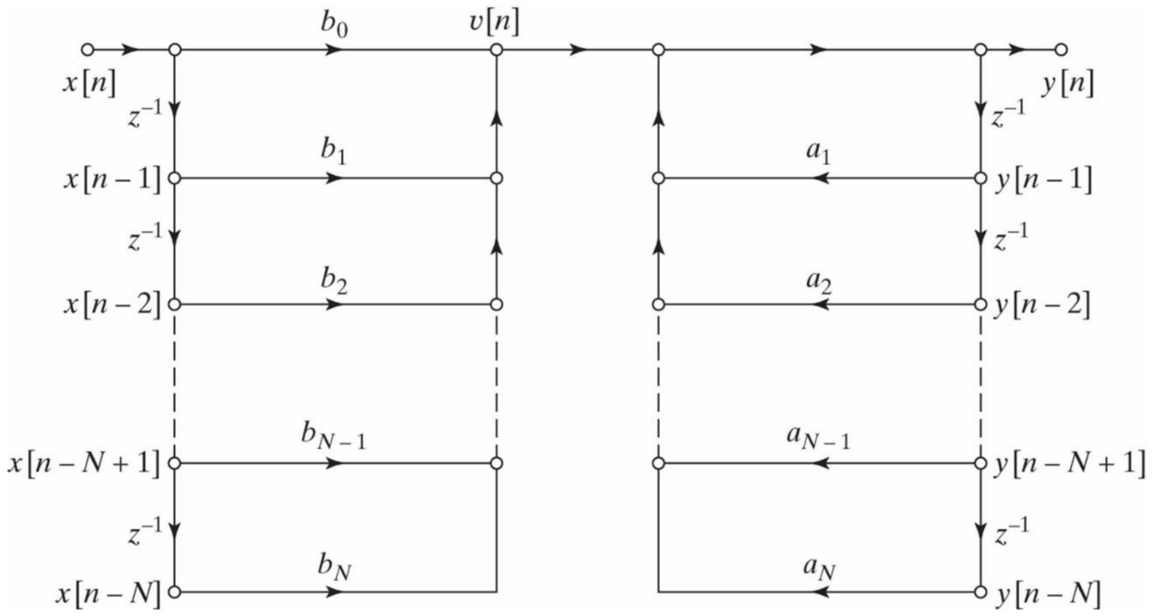
CHAPTER4.3

IIR FILTER REALIZATIONS

- Different forms or structures can implement an IIR filter
  - All are equivalent mathematically (infinite precision)
  - Different practical behavior when considering numerical effects
- Want structures to minimize error

DIRECT FORM I (DFI)

- Straight-forward implementation of diff. eq.
  - b_l - feed forward coefficients
    - From $x(n)$ terms
  - a_l - feedback coefficients
    - From $y(n)$ terms
- Requires $(L + M)$ coefficients and delays



OS 3e

DIRECT FORM II (DFII)

- Notice that we can decompose the transfer function
  - $H(z) = H_1(z)H_2(z)$
  - Section to implement zeros and section to implement poles

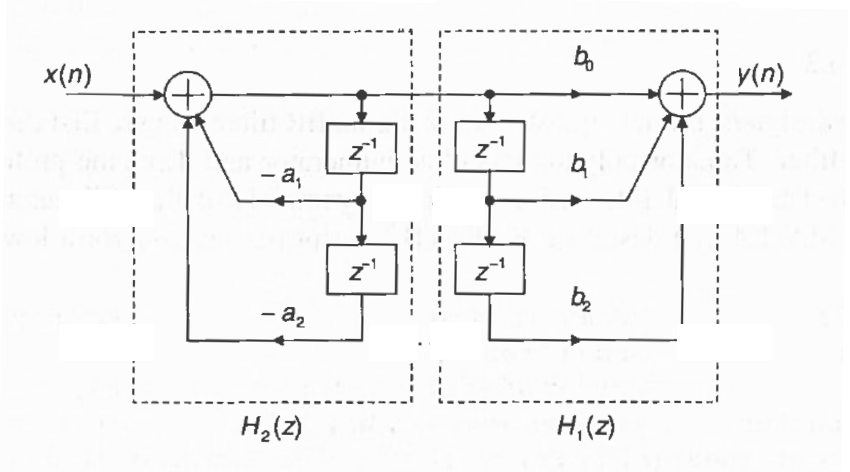
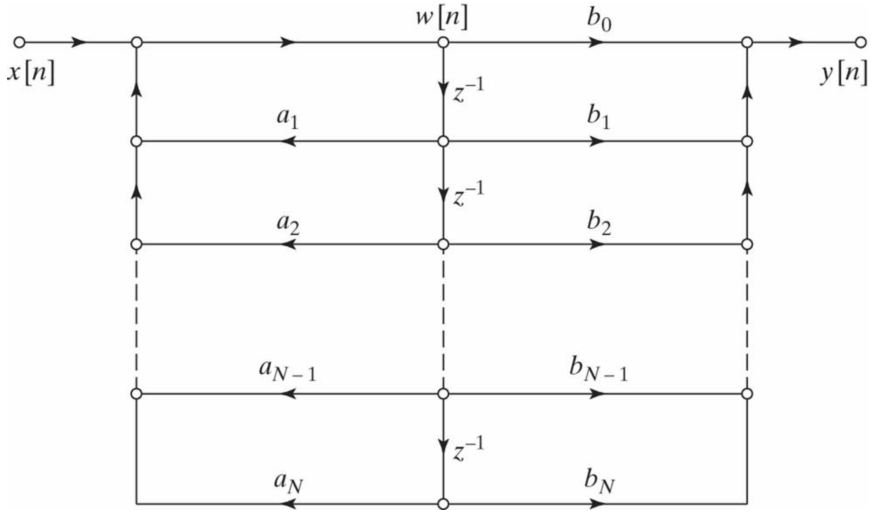


Figure 4.7 Direct-form I realization of second-order IIR filter

- Can switch order of operations
  - $H(z) = H_2(z)H_1(p)$
  - This allows sharing of delays and saving in memory



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CASCADE (FACTORED) FORM

- Factor transfer function and decompose into smaller sub-systems
  - $H(z) = H_1(z)H_2(z)\ldots H_K(z)$
- Make each subsystem second order
  - Complex conjugate roots have real coefficients
  - Limit the order of subsystem (numerical effects)
    - Effects limited to single subsystem stage
    - Change in a single coefficient affects all poles in DF
- Preferred over DF because of numerical stability

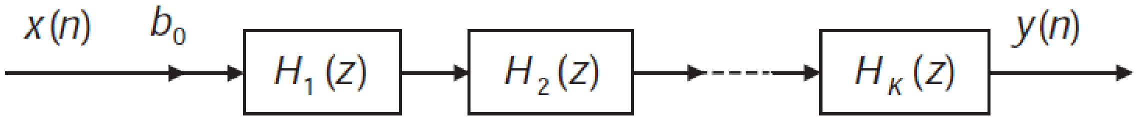
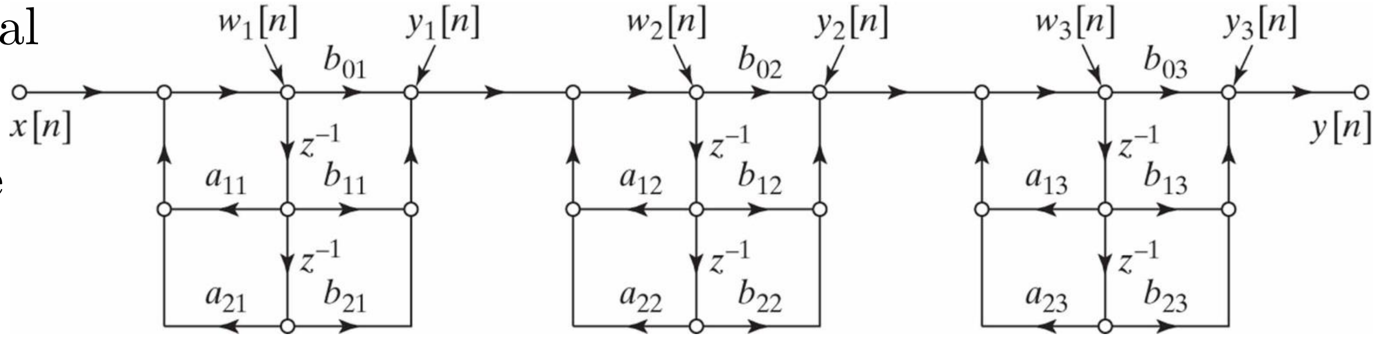


Figure 4.10 Cascade realization of digital filter



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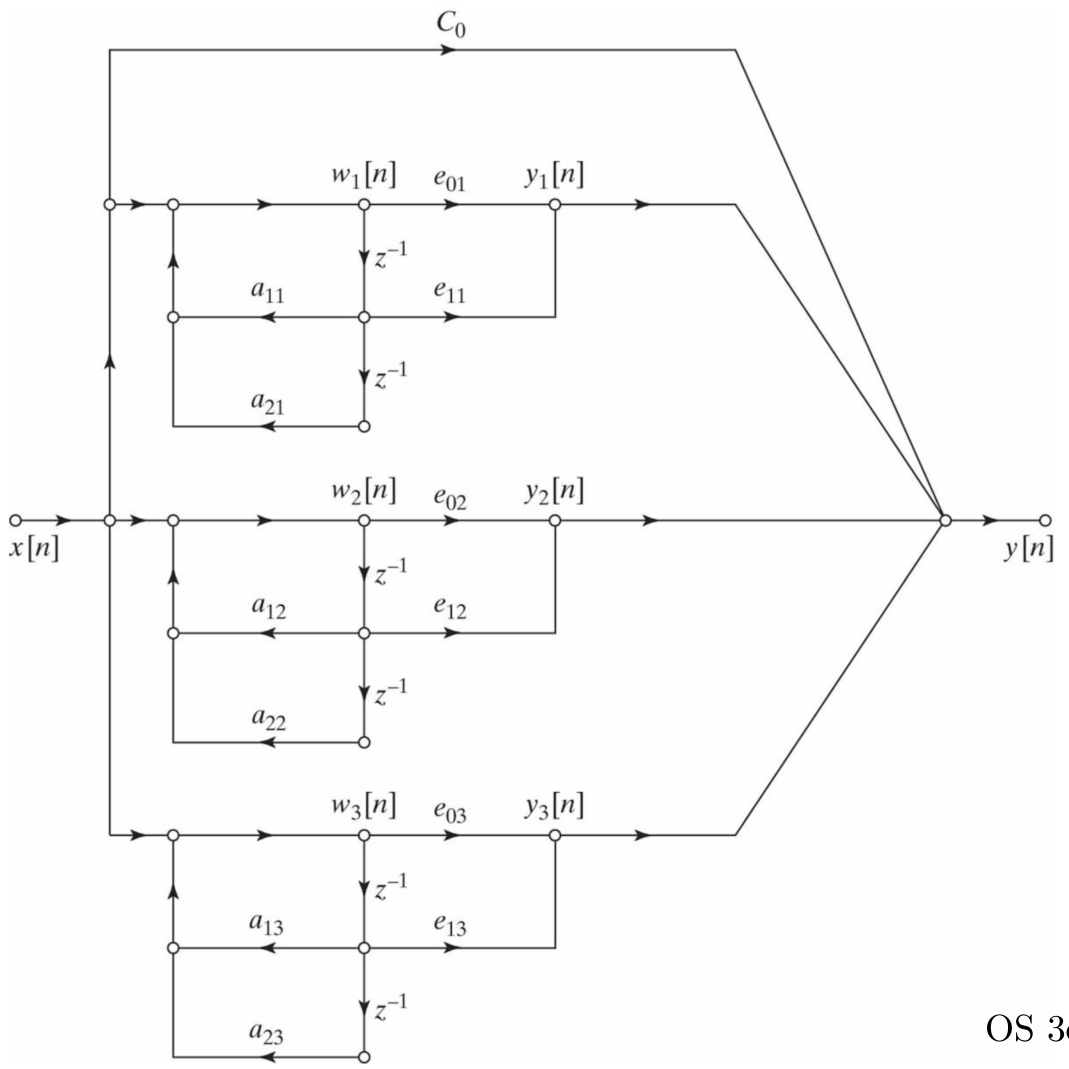
PARALLEL (PARTIAL FRACTION) FORM

- Decompose transfer function using a partial fraction expansion
  - $H(z) = H_1(z) + H_2(z) + \ \ldots + H_K(z)$
  - $H_k(z) = \frac{b_{0k}+b_{1k}z^{-1}}{1+a_{1k}z^{-1}+a_{2k}z^{-2}}$
- Be sure to remember that PFE requires numerator order less than denominator
  - Use polynomial long division



OS 3e

DESIGN OF IIR FILTERS USING MATLAB

CHAPTER4.4

MATLAB FILTER DESIGN

- Realization tools:
- Finding polynomial roots
  - roots.m
  - tf2zp.m
- Cascade form
  - $H(z) = G\prod_{k=1}^{K} \frac{b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}}{1+a_{1k}z^{-1}+a_{2k}z^{-2}}$
  - zp2sos.m
- Parallel form
  - Residuez.m

- Filter design tools:
- Order estimation tool
  - butterord.m
- Coefficient tool
  - butter.m
- Frequency transforms
  - lp2hp.m, lp2bp.m, lp2bs.m

- Useful exploration tool
  - fvtool.m
- Useful design tool
  - fdatool.m
- Useful processing tool
  - sptool.m

IMPLEMENTATION CONSIDERATIONS

CHAPTER4.5

STABILITY

- (Causal) IIR filters are stable if all poles are within the unit circle
  - $|p_m| < 1$
  - We will not consider marginally stable (single pole on unit circle)
- Consider poles of 2nd order filter (used in cascade and parallel forms)
  - $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$
- Factor
  - $A(z) = (1 - p_1 z^{-1})(1 - p_2 z^{-1})$
  - $A(z) = 1 - (p_1 + p_2)z^{-1} + p_1 p_2 z^{-2}$
- Because poles must be inside the unit circle
  - $|a_2| = |p_1 p_2| < 1$
  - $|a_1| < 1 + a_2$

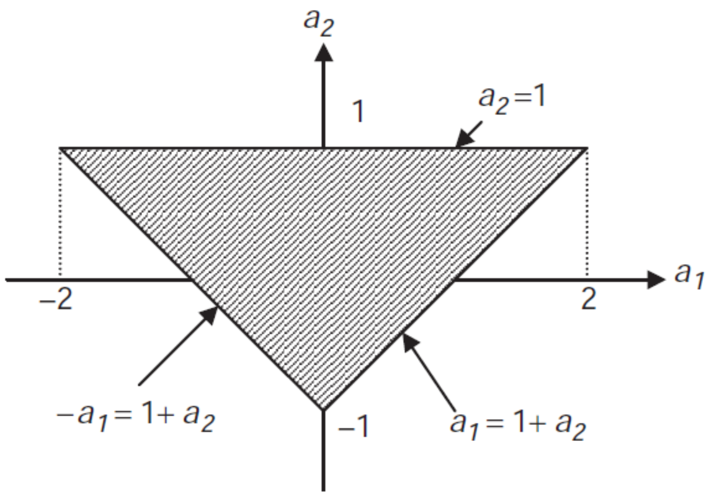


Figure 4.15 Region of coefficient values for the stable second-order IIR filters

COEFFICIENT QUANTIZATION

- Using fixed word lengths results in a quantized approximation of a filter
  - $H'(z) = \frac{\sum_{k=0}^{L-1} b'_k z^{-k}}{1+\sum_{k=1}^{M} a'_k z^{-k}}$
- This can cause a mismatch from desired system $H(z)$
- Poles that are close to the unit circle may move outside and cause instability
  - This is exacerbated with higher order systems

ROUNDING EFFECTS

- Using B bit architecture, products require $2B$ bits
  - Must be rounded into smaller B bit container
- This results in noise error terms
  - Can be simply modeled as additive term
- The order of cascade sections influences power of noise at output
  - How should sections be paired and ordered?
- Need to optimize SQNR
  - Trade-off with probability of arithmetic overflow
  - Need to use scaling factors to prevent overflow
  - Optimality when signal level is maximized without overflow

CASCADE ORDERING AND PAIRING

- Good results are obtained using simple rules
- Cascade ordering and pairing algorithm:
- Pair pole closest to unit circle with zero that is closest in z-plane
  - Minimize the chance of overflow
- Apply 1 repeatedly until all poles and zeros are paired
- Resulting 2nd-order sections can be ordered in two alternative ways
  - Increasing closeness to unit circle
  - Decreasing closeness to unit circle

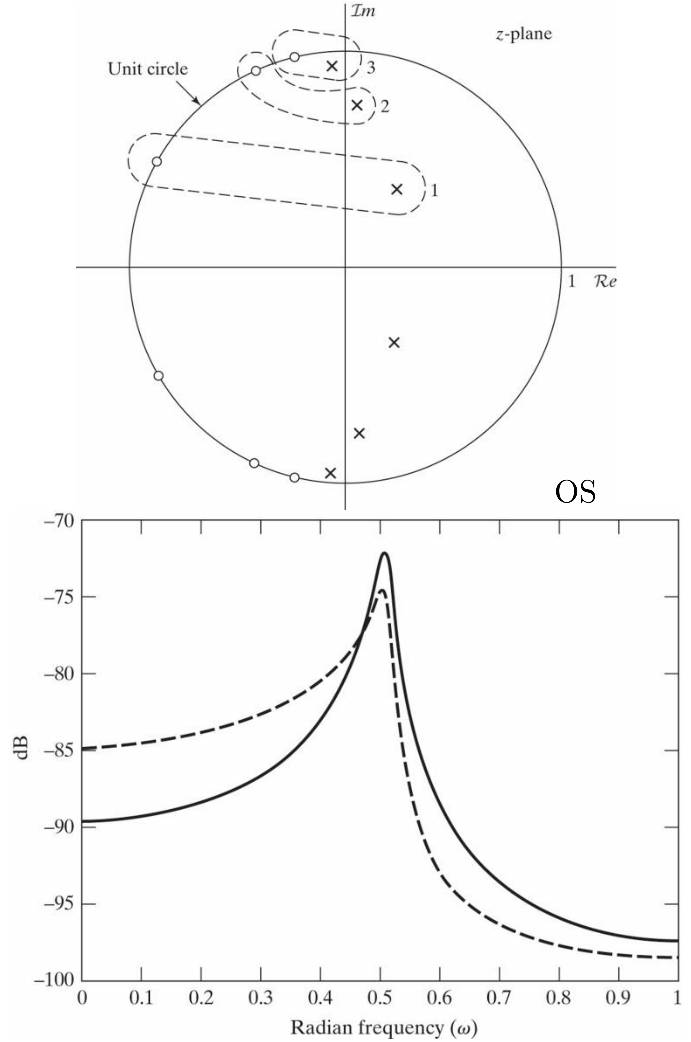


Figure 6.67 Output noise power spectrum for 123 ordering (solid line) and 321 ordering (dashed line) of 2nd-order sections.

PRACTICAL APPLICATIONS

CHAPTER4.6

RECURSIVE RESONATOR

- Filter with frequency response dominated at a single peak
  - Use complex-conjugate pole pair inside unit circle
- $H(z) = \dfrac{A}{(1-r_p e^{j\omega_0} z^{-1})(1-r_p e^{-j\omega_0} z^{-1})}$
- $H(z) = \dfrac{A}{1-2r_p \cos(\omega_0) z^{-1} + r_p^2 z^{-2}}$
  - A – normalization constant for unity gain at ω_0
  - $0 < r_p < 1$
- Close to unit circle
  - Bandwidth $\cong 2(1-r_p)$
  - Closer to $r_p = 1$, more peaked

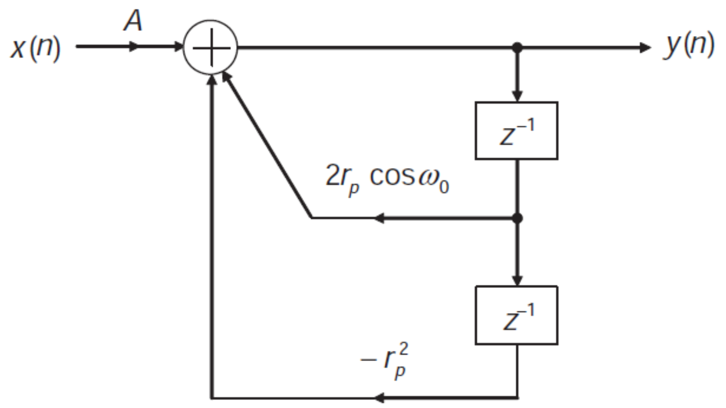
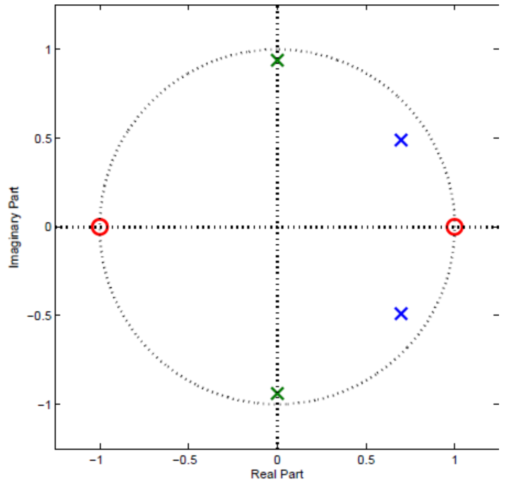
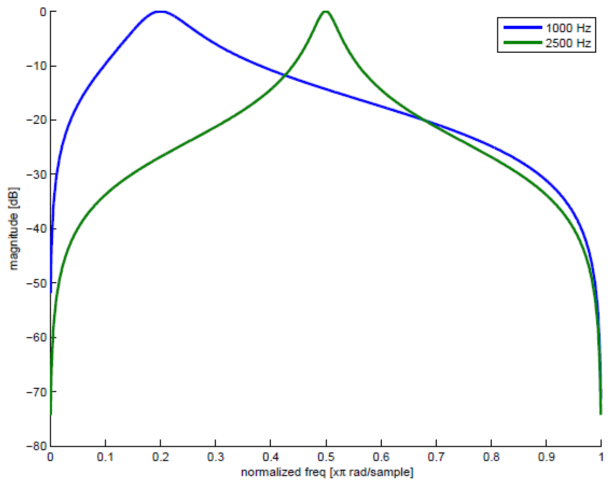


Figure 4.17 Signal-flow diagram of the second-order resonator filter





PARAMETRIC EQUALIZER

- Add nearby zeros to the resonator
  - At same angle as poles ω_0 with similar radius
- Pole and zero counter balance one another
- $r_z < r_p$
  - Pole dominates because it is closer to unit circle
  - Generates peak at $\omega = \omega_0$ → Provides boost to freq
- $r_z > r_p$
  - Zero dominates pole
  - Generates dip at $\omega = \omega_0$ → Cuts freq
- Bandwidth still determined by r_p

- Ex 4.18
  - Create equalizer by changing gain at given frequency