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EE482: Digital Signal Processing Applications

Adaptive Filtering

http://www.ee.unlv.edu/~b1morris/ee482/

Outline

- Random Processes
- Adaptive Filters
- LMS Algorithm

Adaptive Filtering

- FIR and IIR filters are designed for linear time-invariant signals

- How can we handle signals when the characteristics are unknown or changing?

- Need ways to update filter coefficients automatically and continually
  - Track time-varying signals and systems

Random Processes

- Real-world signals are time varying and have randomness in nature
  - E.g. speech, music, noise
- Need to characterize a signal even if full deterministic mathematical definition does not exist
- Random process – sequence of random variables

Autocorrelation

- Specifies statistical relationship of signal at different time lags $(n-k)$
  - $r_{xx}(n,k) = E[x(n)x(k)]$
  - Similarity of observations as a function of the time lag between them
- Mathematical tool for detecting signals
  - Repeating patterns (noise in sinusoid)
  - Measuring time-delay between signals
    - Radar, sonar, lidar
  - Estimation of impulse response
  - Etc.

Wide Sense Stationary (WSS) Process

- Random process statistics do not change with time
- Mean independent of time
  - $E[x(n)] = m_x$
- Autocorrelation only depends only on time lag
  - $r_{xx}(k) = E[x(n+k)x(n)]$
- WSS autocorrelation properties
  - Even function
    - $r_{xx}(-k) = r_{xx}(k)$
  - Bounded by 0 time lag
    - $|r_{xx}(k)| \leq r_{xx}(0) = E[x^2(n)]$
      - Zero mean process: $E[x^2(n)] = \sigma_x^2$
- Cross-correlation
  - $r_{xy}(k) = E[x(n+k)y(n)]$

Expected Value

- Value of random variable "expected" if random variable process repeated infinite number of times
  - Weighted average of all possible values
- Expectation operator
  - $E[.] = \int_{-\infty}^{\infty} . f(x) dx$
  - $f(x)$ – probability density function of random variable X

White Noise

- $v(n)$ with zero mean and variance σ_v^2
- Very popular random signal
  - Typical noise model
- Autocorrelation
  - $r_{vv}(k) = \sigma_v^2 \delta(k)$
  - Statistically uncorrelated except at zero time lag
- Power spectrum
  - $P_{vv}(\omega) = \sigma_v^2, \quad |\omega| \leq \pi$
  - Uniformly distributed over entire frequency range

Example 6.2

- Second-order FIR filter with white noise input ($N(0,\sigma^2)$)
  - $y(n) = x(n) + ax(n-1) + bx(n-2)$
- Mean
  - $E[y(n)] = E[x(n) + ax(n-1) + bx(n-2)]$
  - $E[y(n)] = E[x(n)] + aE[x(n-1)] + bE[x(n-2)]$
  - $E[y(n)] = 0 + a \cdot 0 + b \cdot 0 = 0$
- Autocorrelation
  - $r_{yy}(k) = E[y(n+k)y(n)]$
  - $r_{yy}(k) = E\left[\begin{matrix}\big(x(n+k) + ax(n+k-1) + bx(n+k-2)\big) \cdot \\ \big(x(n) + ax(n-1) + bx(n-2)\big)\end{matrix}\right]$
  - $r_{yy}(k) = E[x(n+k)x(n)] + E[ax(n+k)x(n-1)] + \ ...$
  - $r_{yy}(k) = r_{xx}(k) + ar_{xx}(k-1) + \cdots$
  - $r_{yy}(k) = \begin{cases} (1+a^2+b^2)\sigma_x^2 & k = 0 \\ (a+ab)\sigma_x^2 & k = \pm 1 \\ b\sigma_x^2 & k = \pm 2 \\ 0 & else \end{cases}$

Practical Estimation

- Practical applications have finite length sequences
- Sample mean
  - $\overline{m_x} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$
- Sample autocorrelation
  - $\overline{r_{xx}}(k) = \frac{1}{N-k}\sum_{n-0}^{N-k-1} x(n+k)x(n)$
  - Only produces a good estimate of lags $< 10\%$ of N

- Use Matlab (`mean.m`, `xcorr.m`, etc.) to calculate

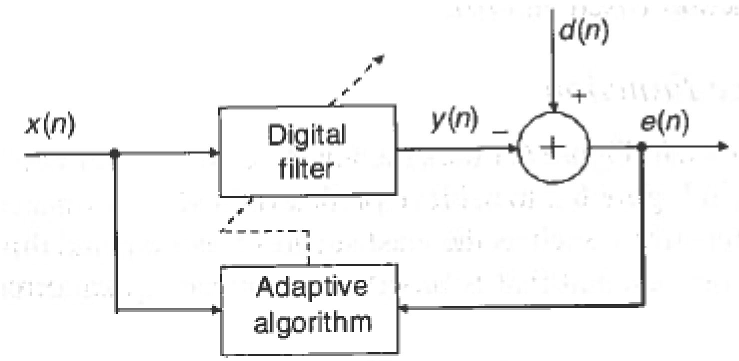
Adaptive Filters

- Signal characteristics in practical applications are time varying and/or unknown
- Must modify filter coefficients adaptively in an automated fashion to meet objectives

- Example: Channel equalization
  - High-speed data communication via media channel (e.g. wireless network)
  - Channel equalization compensates for channel distortion (e.g. path from wifi router and phone)
  - Channel must be continually tracked and characterized to compensate for distortion (e.g. moving around a room)

General Adaptive Filter

- Two components
  - Digital filter – defined by coefficients
  - Adaptive algorithm – automatically update filter coefficients (weights)



- Adaption occurs by comparing filtered signal $y(n)$ with a desired (reference) signal $d(n)$
  - Minimize error $e(n)$ using a cost function (e.g. mean-square error)
  - Continually lower error and get $y(n)$ closer to $d(n)$

FIR Adaptive Filter

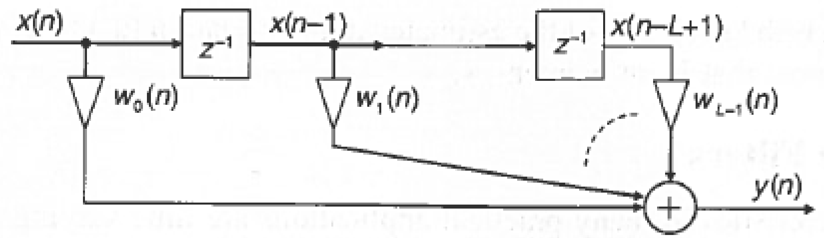


Figure 6.2 Block diagram of time-varying FIR filter for adaptive filtering

- $y(n) = \sum_{l=0}^{L-1} w_l(n)x(n-l)$
  - Notice time-varying weights
- In vector form
  - $y(n) = \boldsymbol{w}^T(n)\boldsymbol{x}(n) = \boldsymbol{x}^T(n)\boldsymbol{w}(n)$
  - $\boldsymbol{x}(n) = [x(n), x(n-1), \ldots, x(n-L+1)]^T$
  - $\boldsymbol{w}(n) = [w_0(n), w_1(n), \ldots, w_{L-1}(n)]^T$
- Error signal
  - $e(n) = d(n) - y(n) = d(n) - \boldsymbol{w}^T(n)\boldsymbol{x}(n)$

Performance Function

- Use mean-square error (MSE) cost function
- $\xi(n) = E[e^2(n)]$
- $\xi(n) = E[d^2(n)] - 2\boldsymbol{p}^T\boldsymbol{w}(n) + \boldsymbol{w}^T(n)\boldsymbol{R}\boldsymbol{w}(n)$
  - $\boldsymbol{p} = E[d(n)\boldsymbol{x}(n)] = [r_{dx}(0), r_{dx}(1), \ldots, r_{dx}(L-1)]^T$
  - $\boldsymbol{R}$ – autocorrelation matrix
    - $\boldsymbol{R} = E[\boldsymbol{x}(n)\boldsymbol{x}^T(n)]$

$$= \begin{bmatrix} r_{xx}(0) & r_{xx}(1) & \ldots & r_{xx}(L-1) \\ r_{xx}(1) & r_{xx}(0) & \ldots & r_{xx}(L-2) \\ \vdots & \ldots & \ddots & \vdots \\ r_{xx}(L-1) & r_{xx}(L-2) & \ldots & r_{xx}(0) \end{bmatrix}, \qquad (6.22)$$

    - Toeplitz matrix – symmetric across main diagonal

Steepest Descent Optimization

- Error function is a quadratic surface
  - $\xi(n) = E[d^2(n)] - 2\boldsymbol{p}^T\boldsymbol{w}(n) + \boldsymbol{w}^T(n)\boldsymbol{R}\boldsymbol{w}(n)$
- Therefore gradient descent search techniques can be used
  - Gradient points in direction of greatest change

- Iterative optimization to "step" toward the bottom of error surface
  - $w(n+1) = w(n) - \frac{\mu}{2}\nabla\xi(n)$

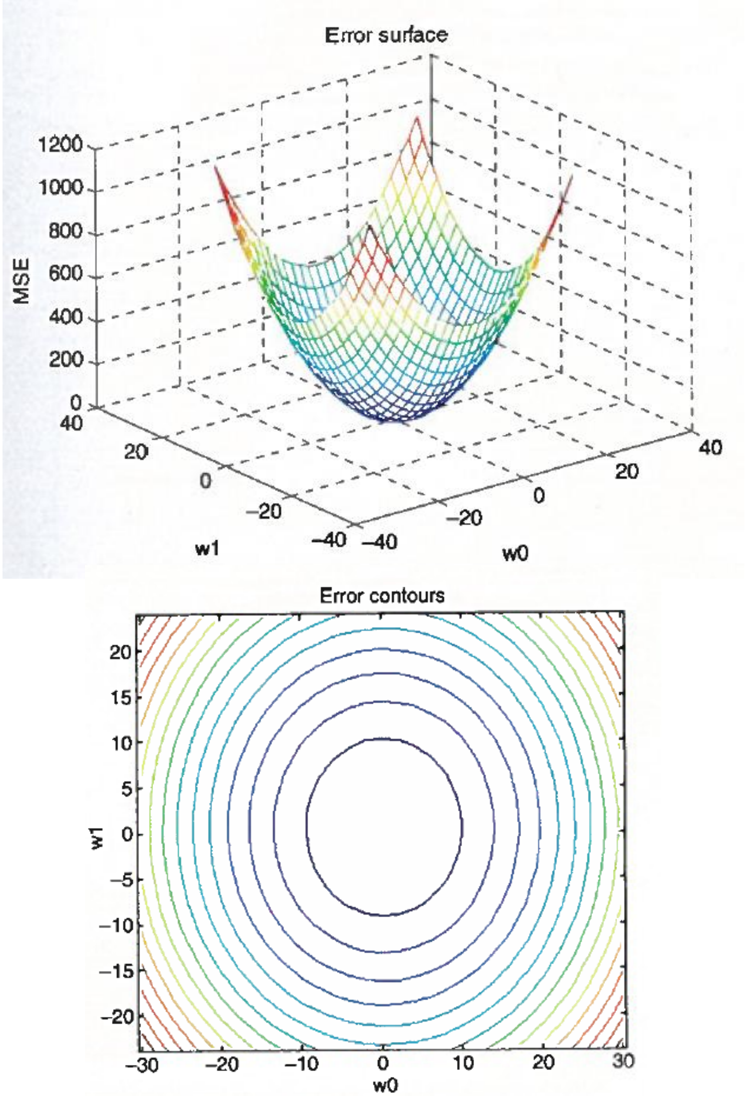


Figure 6.4 Examples of error surface (top) and error contours (bottom), $L = 2$

LMS Algorithm

- Practical applications do not have knowledge of $d(n), x(n)$
  - Cannot directly compute MSE and gradient
  - Stochastic gradient algorithm
- Use instantaneous squared error to estimate MSE
  - $\hat{\xi}(n) = e^2(n)$
- Gradient estimate
  - $\nabla\hat{\xi}(n) = 2[\nabla e(n)]e(n)$
    - $e(n) = d(n) - w^T(n)x(n)$
  - $\nabla\hat{\xi}(n) = -2x(n)e(n)$
- Steepest descent algorithm
  - $w(n+1) = w(n) + \mu x(n)e(n)$

- LMS Steps

1. Set $L, \mu,$ and $\boldsymbol{w}(0)$
   - L – filter length
   - μ – step size (small e.g. 0.01)
   - $\boldsymbol{w}(0)$ – initial filter weights
2. Compute filter output
   - $y(n) = \boldsymbol{w}^T(n)\boldsymbol{x}(n)$
3. Compute error signal
   - $e(n) = d(n) - y(n)$
4. Update weight vector
   - $w_l(n+1) = w_l(n) + \mu x(n-l)e(n),$
     $l = 0,1, \ldots L-1$

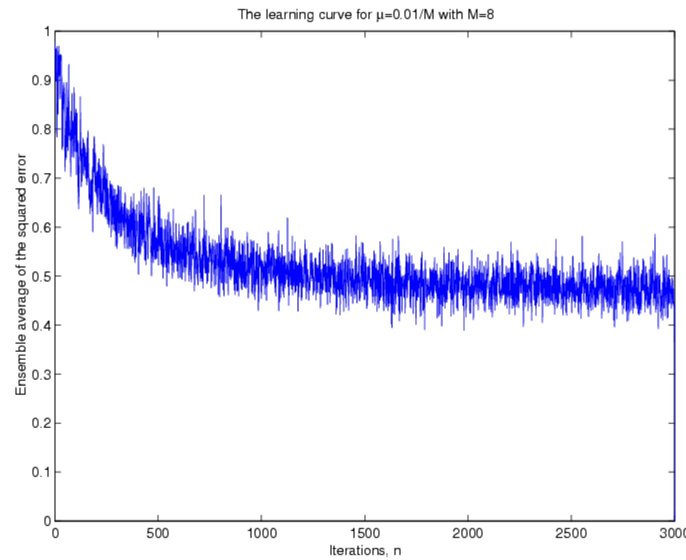
- Notice this requires a reference signal

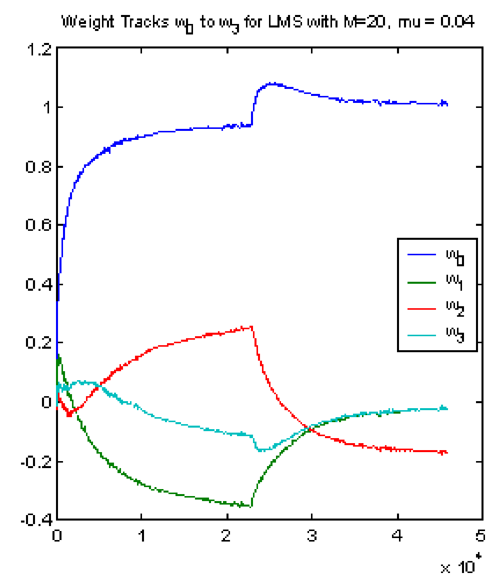
LMS Stability

- Convergence of LMS algorithm
  - $0 < \mu < 2/\lambda_{max}$
    - λ_{max} - largest eigenvalue of autocorrelation matrix $\boldsymbol{R}$
    - Not easy to compute eigenvalues
- Eigenvalue approximation
  - $0 < \mu < 2/LP_x$
    - L – length of data window, filter length
    - $P_x = r_{xx}(0) = E[x^2(n)]$
- Step size is inversely proportional to filter length
  - Smaller μ for higher order filters
- Step size inversely proportional to input signal power
  - Larger μ for lower power signal

Convergence Speed

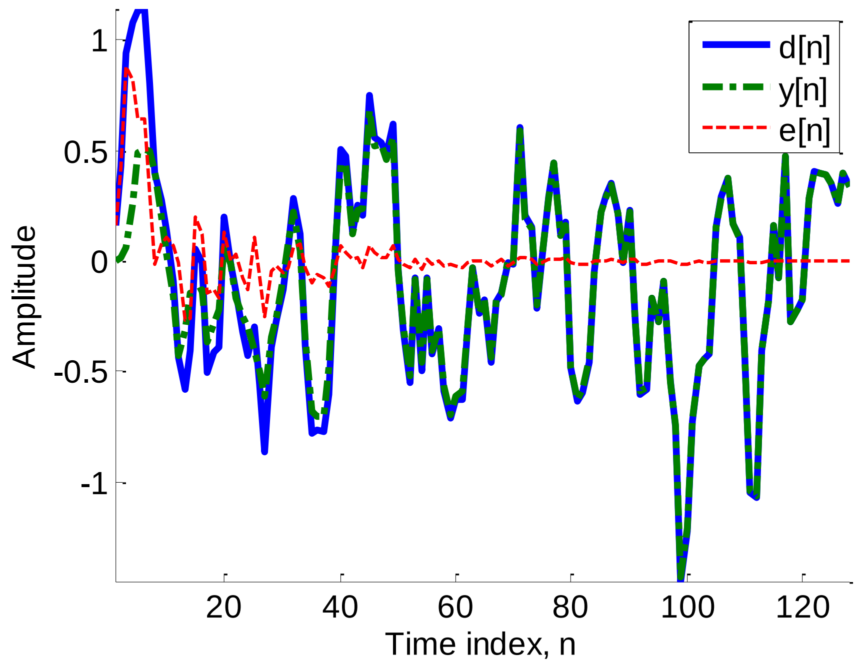
- Convergence of filter weights is defined by the time τ_{MSE} to go from initial MSE to min
  - Plot of MSE vs. time is known as the learning curve
- Convergence time related to the minimum eigenvalue of $\boldsymbol{R}$
  - $\tau_{MSE} \cong \dfrac{1}{\mu\lambda_{min}}$
    - Smaller step size results in longer convergence time
- In practice, weights will not converge to a fixed optimum value but will vary around it





Example 6.7

- `sd = 12357; rng(sd);          % Set seed value`
- `x  = randn(1,128);            % Reference signal x(n)`
- `b  = [0.1,0.2,0.4,0.2,0.1];   % An FIR filter to be identified`
- `d  = filter(b,1,x);           % Desired signal d(n)`
- `mu = 0.05;                    % Step size mu`
- `h = adaptfilt.lms(5,mu);      % LMS algorithm`
- `[y,e] = filter(h,x,d);        % Adaptive filtering`
- `n = 1:128;`
- `h1=figure;`
- `hold all;`
- `plot(n,d,'-','linewidth', 3);`
- `plot(n,y,'-.', 'linewidth', 3);`
- `plot(n,e,'--', 'linewidth', 2);`
- `axis([1 128 -inf inf]);`
- `xlabel('Time index, n');`
- `ylabel('Amplitude');`
- `legend('d[n]', 'y[n]', 'e[n]');`
-
- `[b; h.coefficients]`



- Coefficients
- $b = [0.1000 \quad 0.2000 \quad 0.4000 \quad 0.2000 \quad 0.1000]$
- $w =$ $[0.1005 \quad 0.1999 \quad 0.3996 \quad 0.1995 \quad 0.0996]$

Practical Applications

- Four classes of adaptive filtering applications
  - System identification
  - Prediction
  - Noise cancellation
  - Inverse modeling

- Differences based on configuration of control signals $x(n), d(n), y(n), e(n)$

System Identification

- Given an unknown system, try to determine (identify) coefficients

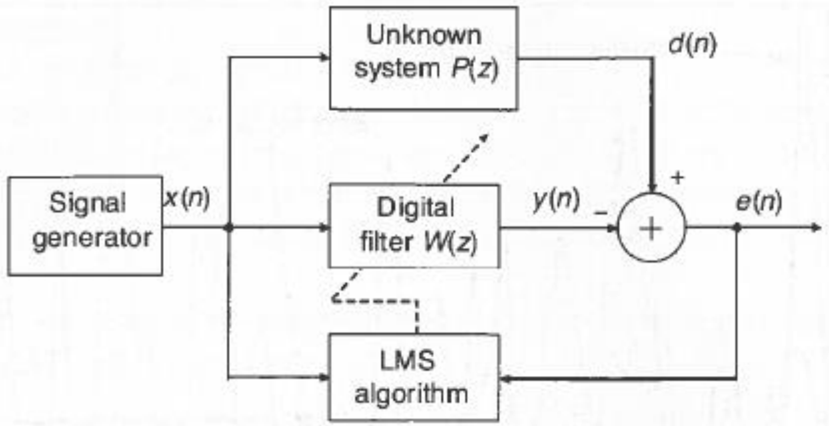


Figure 6.7 Adaptive system identification using the LMS algorithm

- Excite unknown system and adaptive system with same input
  - Input signal: white noise
  - Reference signal: output of unknown system
  - Error is difference between adaptive filter and the output of unknown system

Prediction

- Linear predictor estimates signal values at future times

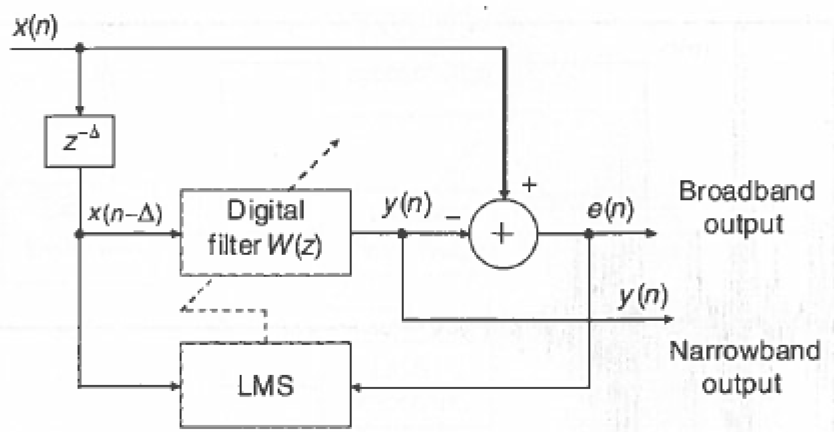
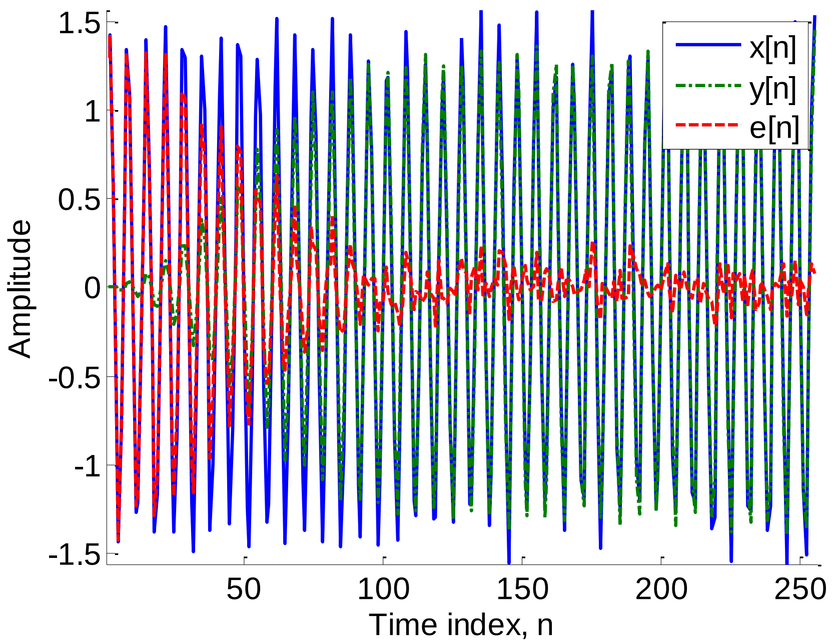


Figure 6.9 Adaptive predictor with the LMS algorithm

- Reference signal: signal of interest
- Input signal: delayed reference signal
- Error is difference between current sample and predicted sample (using past samples)
  - Leverage correlation between samples
- Broadband output: noise component
- Narrowband output: signal of interest (high correlation)

Example 6.9

```
• Fs = 1000;
• f0 = 150;
• L =64;
• N=256;
• A=sqrt(2);
• w0=2*pi*f0/Fs;
• n = [0:N-1];
• sn = A*sin(w0*n);
• vn = 0.1*(rand(1,N)-0.5)*sqrt(12)
• x = sn+vn
• d  = [0, x(2:256)];
• mu = 0.001;
• h = adaptfilt.lms(L,mu);
• [y,e] = filter(h,x,d)
• h1=figure;
• hold all;
• plot(n,x,'-','linewidth', 2);
• plot(n,y,'-.', 'linewidth', 2);
• plot(n,e,'--', 'linewidth', 2);
• axis([1 N -inf inf]);
• xlabel('Time index, n');
• ylabel('Amplitude');
• legend('x[n]', 'y[n]', 'e[n]');
```



Noise Cancellation

- Remove (cancel) noise components embedded in a primary signal
  - E.g. background noise in speech signal

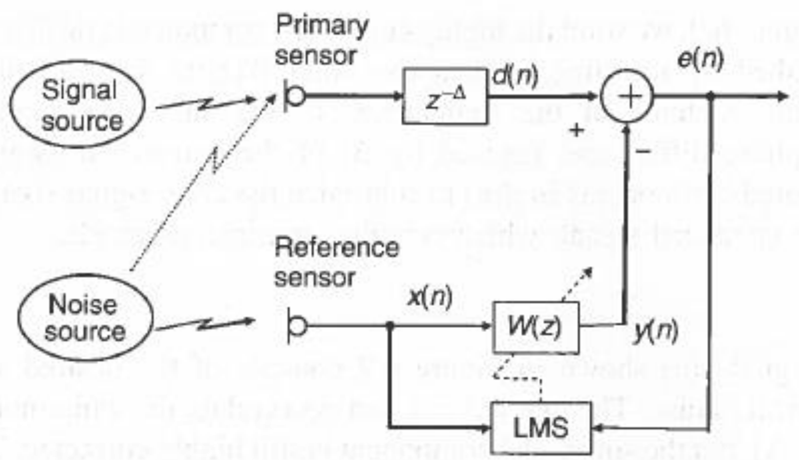


Figure 6.11 Basic concept of adaptive noise canceling

- Flip idea of reference and input signals
  - Reference signal: primary signal + noise
    - Close to primary source
  - Input signal: noise signal
    - Far from primary source to measure noise
  - Adaptive filter tracks correlated noise
    - Error signal is the desired cleaned primary signal

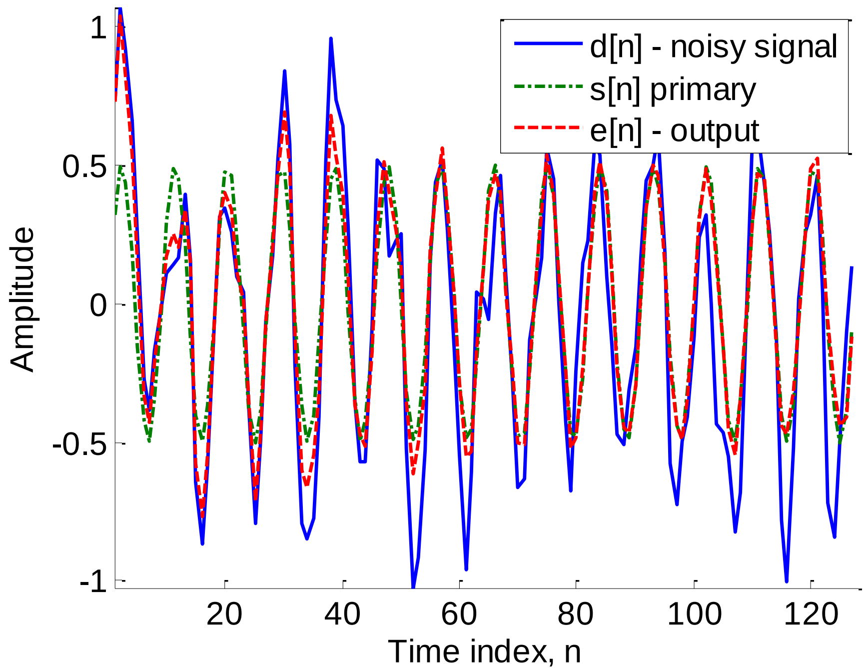
Example 6.10

```
Fs = 1000;
f0 = 110;
L = 3;
N = 128;
w0 = 2*pi*f0/Fs;
pz = [0.1, 0.3, 0.2];       % Define noise path
n = [0:N-1];                % Time index
sd = 12357; rng(sd);        % Set seed value

sn = 0.5*sin(w0*n);         % Sine sequence
xn = 2.5*(rand(1,N)-0.5);   % Zero-mean white
noise
xpn = filter(pz, 1, xn);    % Generate x'(n)
dn = sn+xpn;                % Sinewave embedded
in white noise

mu = 0.025;                 % Step size mu
h = adaptfilt.lms(L,mu);    % LMS algorithm
[y,e] = filter(h,xn,dn);    % Adaptive
filtering


h1=figure;
hold all;
plot(n,dn,'-','linewidth', 2);
plot(n,sn,'-.', 'linewidth', 2);
plot(n,e,'--', 'linewidth', 2);
axis([1 N -inf inf]);
xlabel('Time index, n');
ylabel('Amplitude');
legend('d[n] - noisy signal', 's[n] primary',
'e[n] - output');
```



Inverse Modeling

- Method to estimate the inverse of an unknown system
    - E.g. a communication channel is unknown but its distortion needs to be corrected
- Reference signal: a known training signal
- Input signal: training signal after going through unknown system

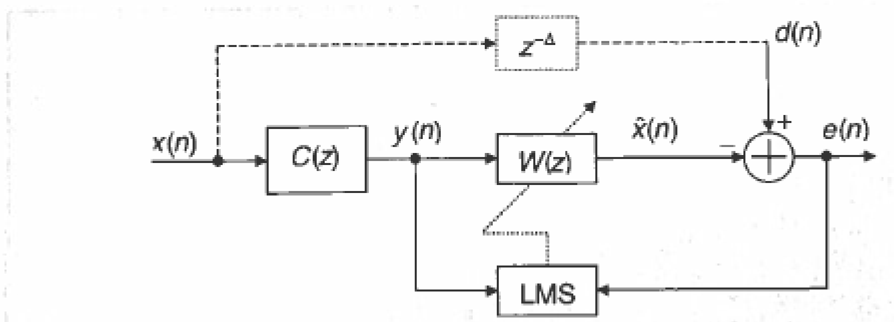


Figure 6.14 An adaptive channel equalizer as an example of inverse modeling