

EE482: Digital Signal Processing Applications

IIR Design

Outline

- Analog Filter Characteristics
- Frequency Transforms
- Design of IIR Filters
- Realizations of IIR Filters
 - Direct, Cascade, Parallel
- Implementation Considerations

IIR Design

- Reuse well studied analog filter design techniques (books and tables for design)
- Need to map between analog design and a digital design
 - Mapping between s-plane and z-plane

Analog Basics

- Laplace transform
 - $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$
- Complex s-plane
 - $s = \sigma + j\Omega$
 - Complex number with σ and Ω real
 - $j\Omega$ – imaginary axis
- Fourier transform for $\sigma = 0$
 - When region of convergence contains the $j\Omega$ axis
- Convolution relationship
 - $y(t) = x(t) * h(t) \rightarrow Y(s) = X(s)H(s)$
 - $H(s) = \frac{Y(s)}{X(s)} = \int_{-\infty}^{\infty} h(t)e^{-st} dt$
- Stability constraint requires poles to be in the left half s-plane

Mapping Properties

- z-transform from Laplace by change of variable
 - $z = e^{sT} = e^{\sigma T} e^{j\Omega T} = |z|e^{j\omega}$
 - $|z| = e^{\sigma T}$, $\omega = \Omega T$
- This mapping is not unique
 - $-\pi/T < \Omega < \pi/T \rightarrow$ unit circle
 - 2π multiples as well

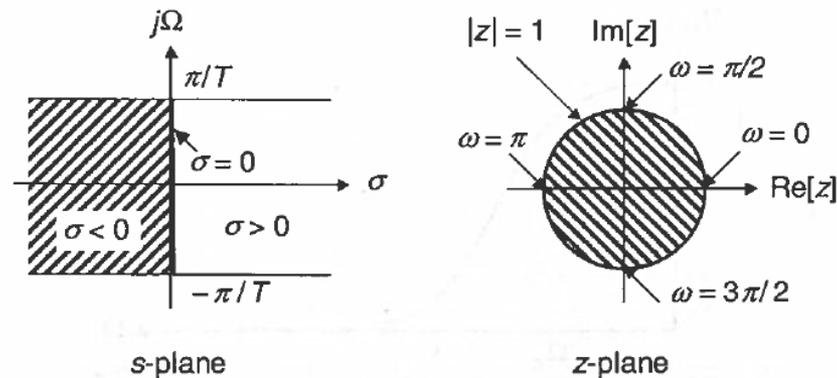


Figure 4.1 Mapping properties between the s-plane and the z-plane

- Left half s-plane mapped inside unit circle
- Right half s-plane mapped outside unit circle

Filter Characteristics

- Designed to meet a given/desired magnitude response
- Trade-off between :
 - Phase response
 - Roll-off rate – how steep is the transition between pass and stopband (transition width)

Butterworth Filter

- All-pole approximation to idea filter
- $|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_p)^{2L}}$
 - $|H(0)| = 1$
 - $|H(\Omega_p)| = 1/\sqrt{2}$
 - -3 dB @ Ω_p
- Has flat magnitude response in pass and stopband (no ripple)
- Slow monotonic transition band
 - Generally needs larger L

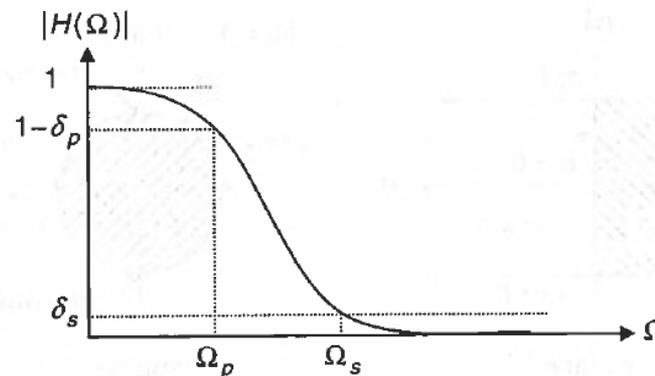


Figure 4.2 Magnitude response of Butterworth lowpass filter

Chebyshev Filter

- Steeper roll-off at cutoff frequency than Butterworth
 - Allows certain number of ripples in either passband or stopband
- Type I – equiripple in passband, monotonic in stopband
 - All-pole filter
- Type II – equiripple in stopband, monotonic in passband
 - Poles and zeros
- Generally better magnitude response than Butterworth but at cost of poorer phase response

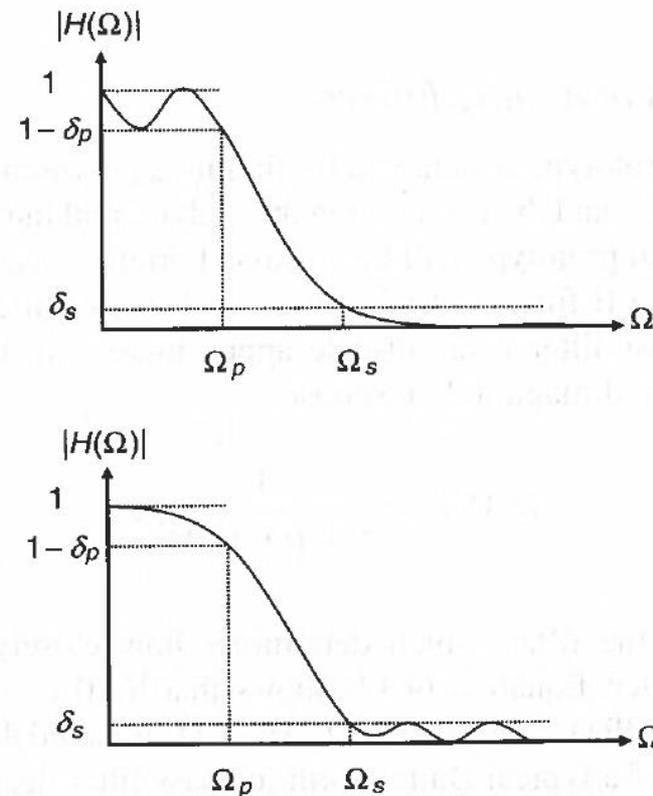


Figure 4.3 Magnitude responses of Chebyshev type I (top) and type II lowpass filters

Elliptic Filter

- Sharpest passband to stopband transition
- Equiripple in both pass and stopbands
- Phase response is highly nonlinear in passband
 - Should only be used in situations where phase is not important to design

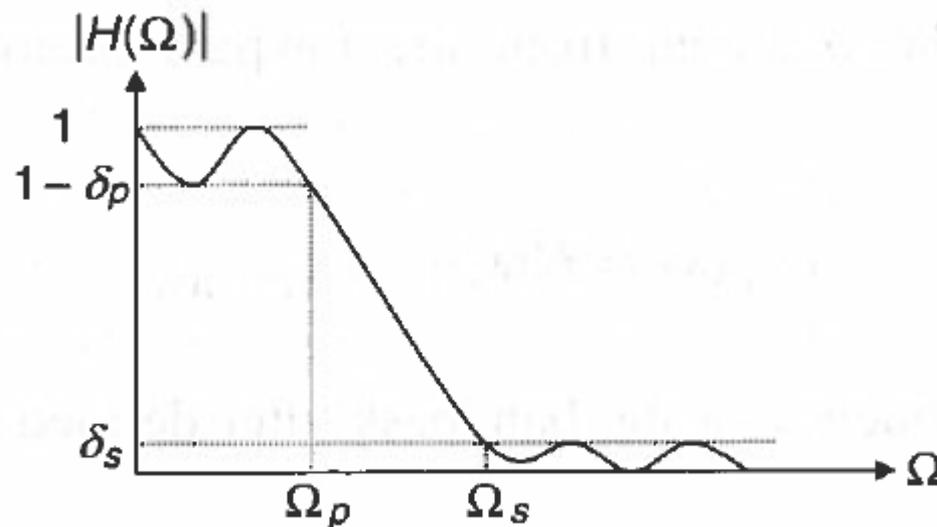


Figure 4.4 Magnitude response of elliptic lowpass filter

Frequency Transforms

OS 3e

- Design lowpass filter and transform from LP to another type (HP, BP, BS)
- Define mapping
- $H(z) = H_{lp}(Z)|_{Z^{-1}=G(z^{-1})}$
 - Replace Z^{-1} in LP filter with $G(z^{-1})$
- θ – frequency in LP
- ω – frequency in new transformed filter

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k}z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k}z^{-2} - \frac{2\alpha}{1+k}z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

IIR Filter Design

- IIR transfer function

$$H(z) = \frac{\sum_{l=0}^{L-1} b_l z^{-l}}{1 + \sum_{l=0}^M a_l z^{-l}}$$

- Need to find coefficients a_l, b_l
 - Impulse invariance – sample impulse response
 - Have to deal with aliasing
 - Bilinear transform
 - Match magnitude response
 - “Warp” frequencies to prevent aliasing

Bilinear Transform Design

- Convert digital filter into an “equivalent” analog filter
 - Use bilinear “warping”
- Design analog filter using IIR design techniques
- Map analog filter into digital
 - Use bilinear transform

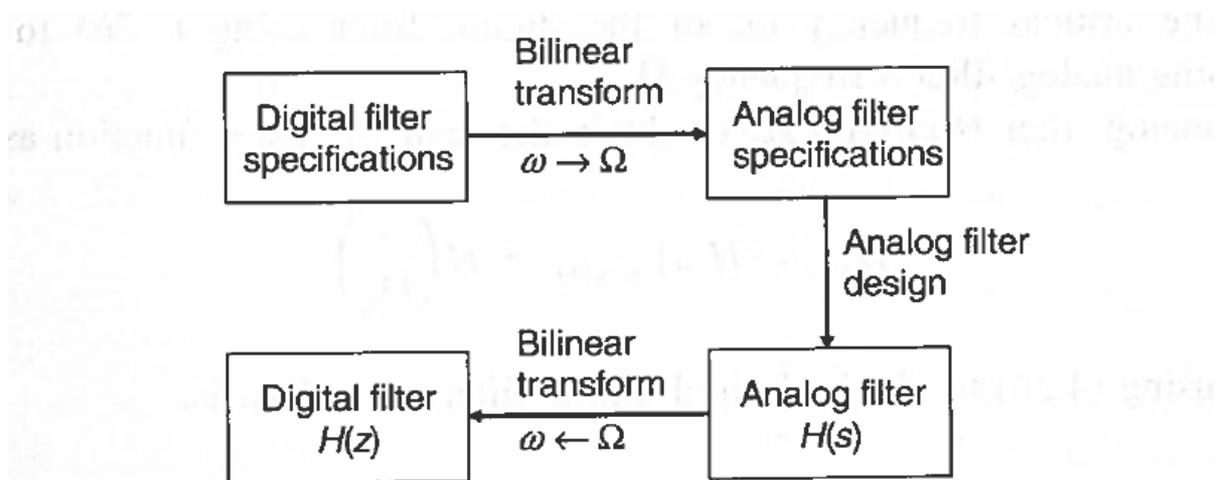


Figure 4.5 Digital IIR filter design using the bilinear transform

Bilinear Transformation

- Mapping from s-plane to z-plane
- $s = \frac{2}{T} \left(\frac{z-1}{z+1} \right) = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)$
- Frequency mapping
 - $\Omega = \frac{2}{T} \tan \left(\frac{\omega}{2} \right)$
 - $\omega = 2 \arctan \left(\frac{\Omega T}{2} \right)$
- Entire $j\omega$ -axis is squished into $[-\pi/T, \pi/T]$ to prevent aliasing
 - Unique mapping
 - Highly non-linear which requires “pre-warp” in design

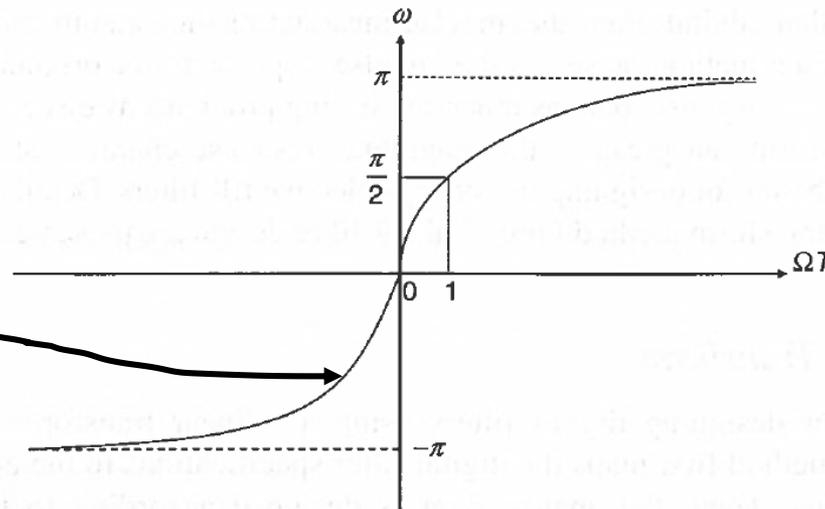
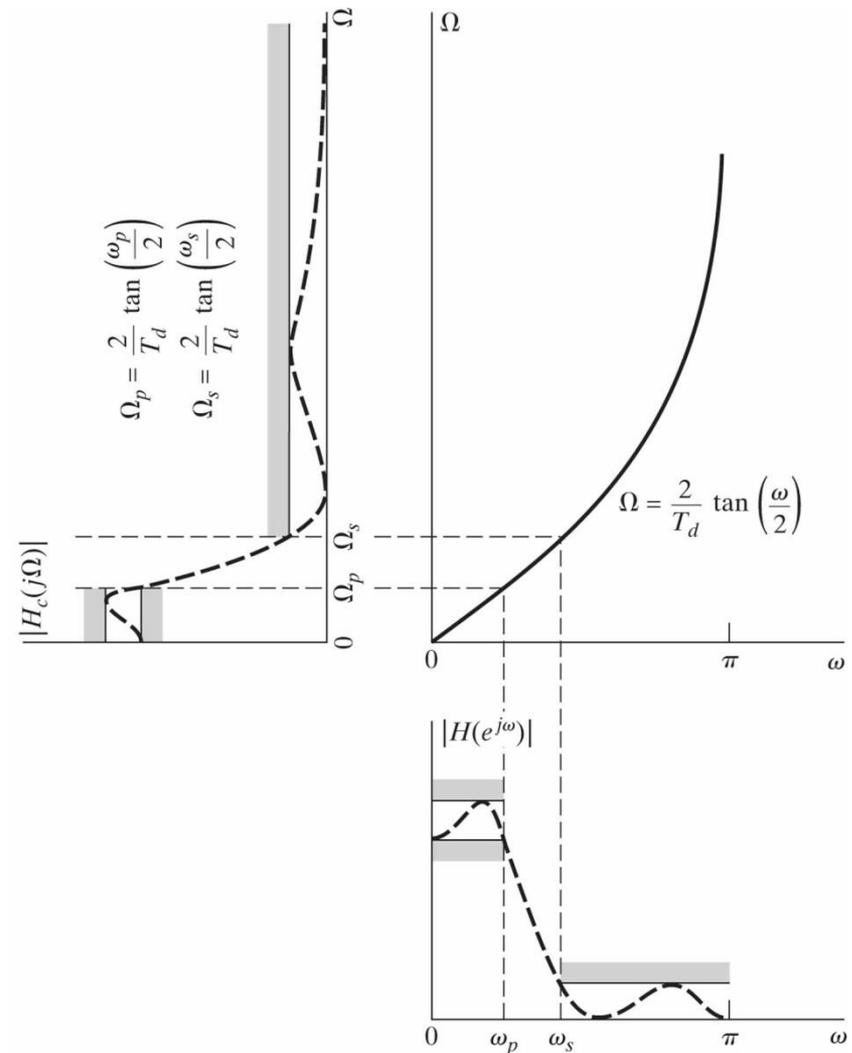


Figure 4.6 Frequency warping of bilinear transform defined by (4.27)

Bilinear Design Steps

1. Convert digital filter into an “equivalent” analog filter
 - Pre-warp using
 - $\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$
2. Design analog filter using IIR design techniques
 - Butterworth, Chebyshev, Elliptical
3. Map analog filter into digital
 - $H(z) = H(s) \Big|_{s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)}$



Bilinear Design Example

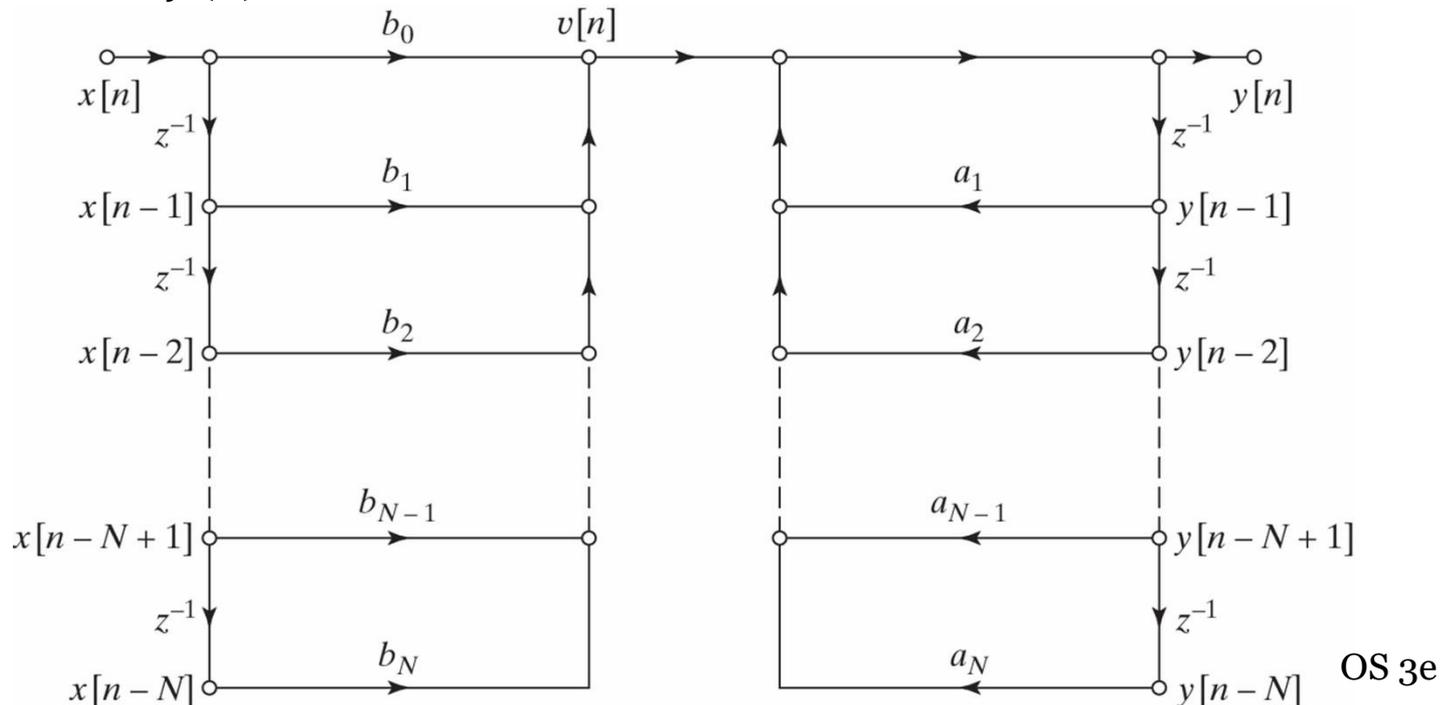
- Example 4.2
- Design filter using bilinear transform
 - $H(s) = 1/(s + 1)$
 - Bandwidth 10000 Hz
 - $f_s = 8000$ Hz
- Parameters
 - $\omega_c = 2\pi(1000/8000) = 0.25\pi$
- 1. Pre-warp
 - $\Omega_c = \frac{2}{T} \tan(0.125\pi) = \frac{0.8284}{T}$
- 2. Scale frequency (normalize scale)
 - $\hat{H}(s) = H\left(\frac{s}{\Omega_c}\right) = \frac{0.8284}{sT + 0.8284}$
- 3. Bilinear transform
 - $H(z) = \frac{0.2929(1+z^{-1})}{1-0.4141z^{-1}}$

IIR Filter Realizations

- Different forms or structures can implement an IIR filter
 - All are equivalent mathematically (infinite precision)
 - Different practical behavior when considering numerical effects
- Want structures to minimize error

Direct Form I

- Straight-forward implementation of diff. eq.
 - b_l - feed forward coefficients
 - From $x(n)$ terms
 - a_l - feedback coefficients
 - From $y(n)$ terms
- Requires $(L + M)$ coefficients and delays



Direct Form II

- Notice that we can decompose the transfer function
 - $H(z) = H_1(z)H_2(z)$
 - Section to implement zeros
 - section to implement poles

- Can switch order of operations
 - $H(z) = H_2(z)H_1(z)$
 - This allows sharing of delays and saving in memory

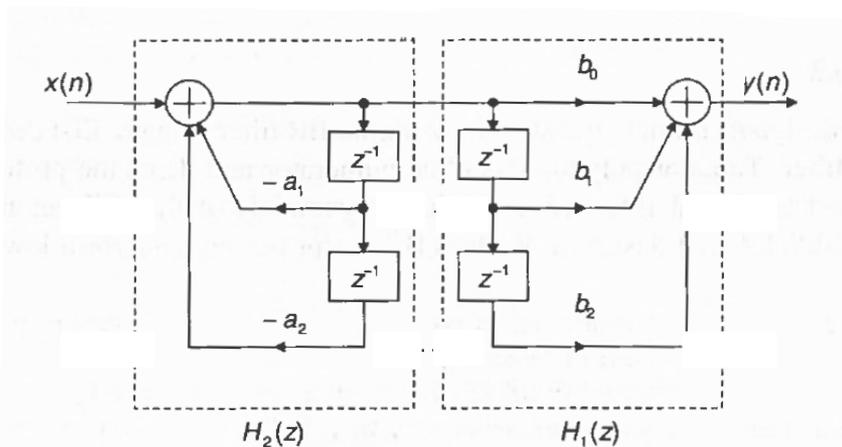
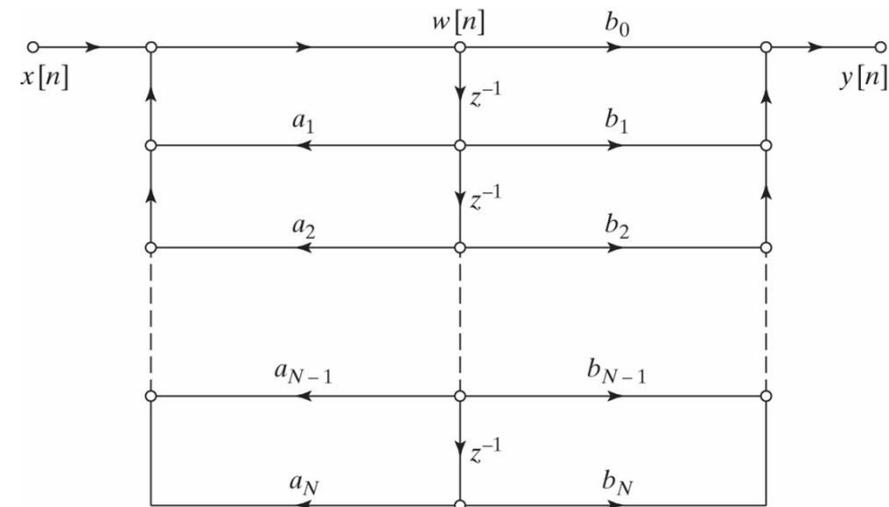


Figure 4.7 Direct-form I realization of second-order IIR filter



Cascade (Factored) Form

- Factor transfer function and decompose into smaller sub-systems
 - $H(z) = H_1(z)H_2(z) \dots H_K(z)$

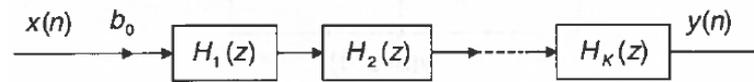
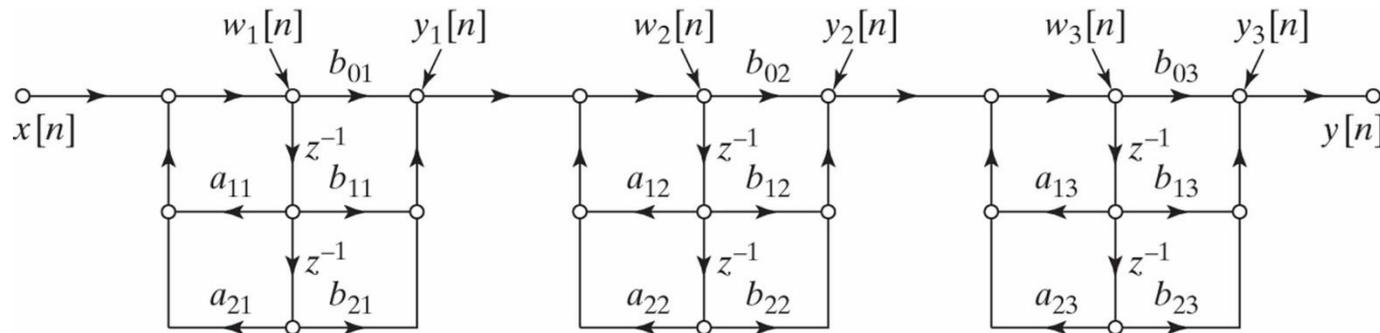


Figure 4.10 Cascade realization of digital filter

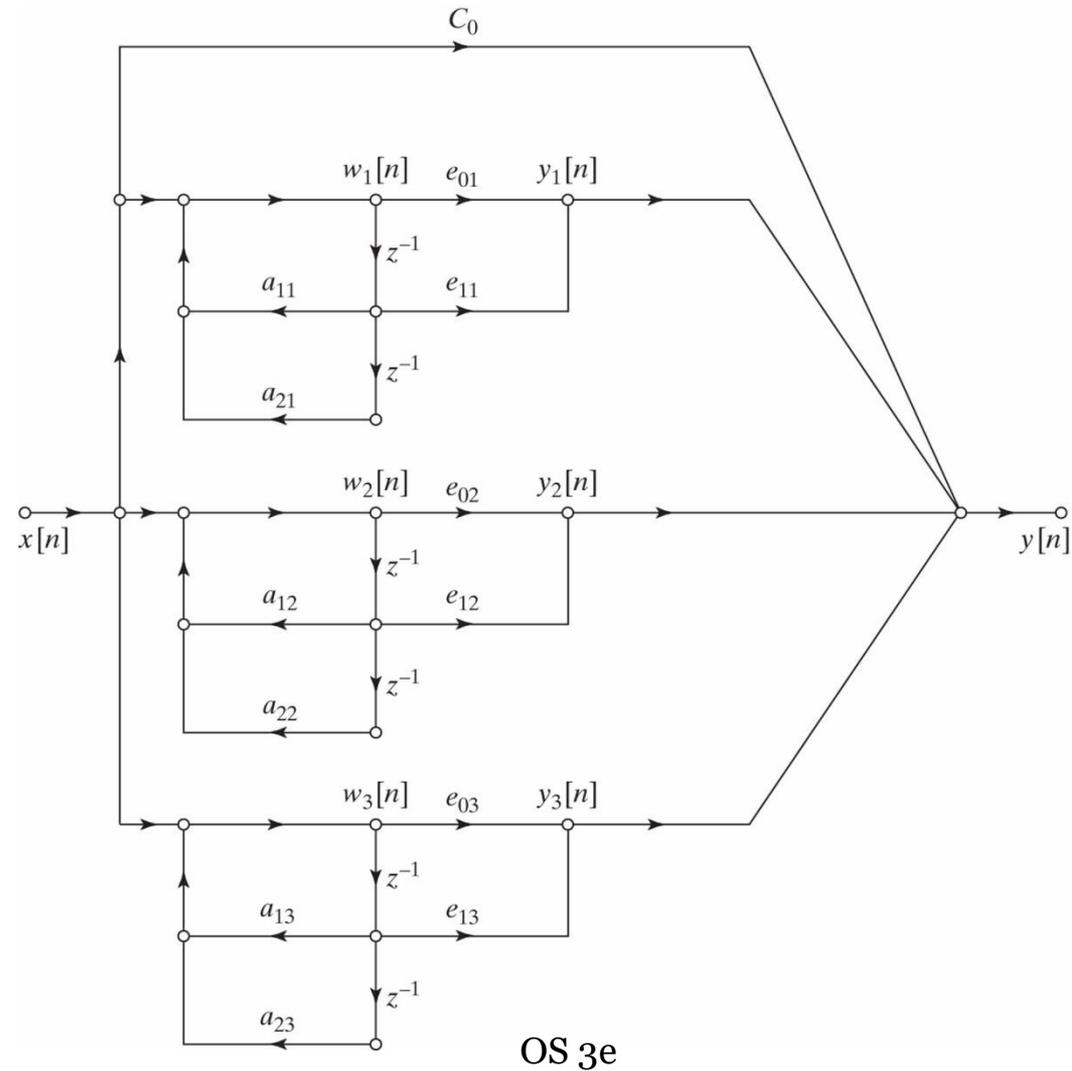
- Make each subsystem second order
 - Complex conjugate roots have real coefficients
 - Limit the order of subsystem (numerical effects)
 - Effects limited to single subsystem stage
 - Change in a single coefficient affects all poles in DF



- Preferred over DF because of numerical stability

Parallel (Partial Fraction) Form

- Decompose transfer function using a partial fraction expansion
 - $H(z) = H_1(z) + H_2(z) + \dots + H_K(z)$
 - $H_k(z) = \frac{b_{0k} + b_{1k}z^{-1}}{1 + a_{1k}z^{-1} + a_{2k}z^{-2}}$
- Be sure to remember that PFE requires numerator order less than denominator
 - Use polynomial long division



Matlab Filter Design

- Realization tools:
- Finding polynomial roots
 - `roots.m`
 - `tf2zp.m`
- Cascade form
 - $H(z) = G \prod_{k=1}^K \frac{b_{0k} + b_{1k}z^{-1} + b_{2k}z^{-2}}{1 + a_{1k}z^{-1} + a_{2k}z^{-2}}$
 - `zp2sos.m`
- Parallel form
 - `Residuez.m`
- Filter design tools:
- Order estimation tool
 - `butterord.m`
- Coefficient tool
 - `butter.m`
- Frequency transforms
 - `lp2hp.m`, `lp2bp.m`, `lp2bs.m`
- Useful exploration tool
 - `fvtool.m`
- Useful design tool
 - `fdatool.m`
- Useful processing tool
 - `sptool.m`

Stability

- (Causal) IIR filters are stable if all poles are within the unit circle
 - $|p_m| < 1$
 - We will not consider marginally stable (single pole on unit circle)
- Consider poles of 2nd order filter (used in cascade and parallel forms)
 - $A(z) = 1 + a_1z^{-1} + a_2z^{-2}$
- Factor
 - $A(z) = (1 - p_1z^{-1})(1 - p_2z^{-1})$
 - $A(z) = 1 - (p_1 + p_2)z^{-1} + p_1p_2z^{-2}$
- Because poles must be inside the unit circle
 - $|a_2| = |p_1p_2| < 1$
 - $|a_1| < 1 + a_2$

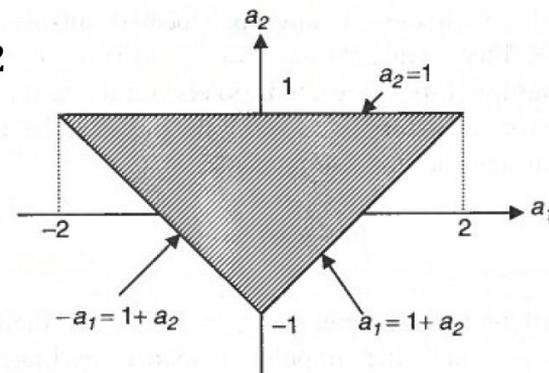


Figure 4.15 Region of coefficient values for the stable second-order IIR filters

Coefficient Quantization

- Using fixed word lengths results in a quantized approximation of a filter
 - $$H'(z) = \frac{\sum_{k=0}^{L-1} b'_k z^{-k}}{1 + \sum_{k=1}^M a'_k z^{-k}}$$
- This can cause a mismatch from desired system $H(z)$
- Poles that are close to the unit circle may move outside and cause instability
 - This is exacerbated with higher order systems

Rounding Effects

- Using B bit architecture, products require $2B$ bits
 - Must be rounded into smaller B bit container
- This results in noise error terms
 - Can be simply modeled as additive term
- The order of cascade sections influences power of noise at output
 - How should sections be paired and ordered?
- Need to optimize SQNR
 - Trade-off with probability of arithmetic overflow
 - Need to use scaling factors to prevent overflow
 - Optimality when signal level is maximized without overflow

Cascade Ordering and Pairing

- Good results are obtained using simple rules
- Cascade ordering and pairing algorithm
 1. Pair pole closest to unit circle with zero that is closest in z-plane
 - Minimize the chance of overflow
 2. Apply 1 repeatedly until all poles and zeros are paired
 3. Resulting 2nd-order sections can be ordered in two alternative ways
 - Increasing closeness to unit circle
 - Decreasing closeness to unit circle

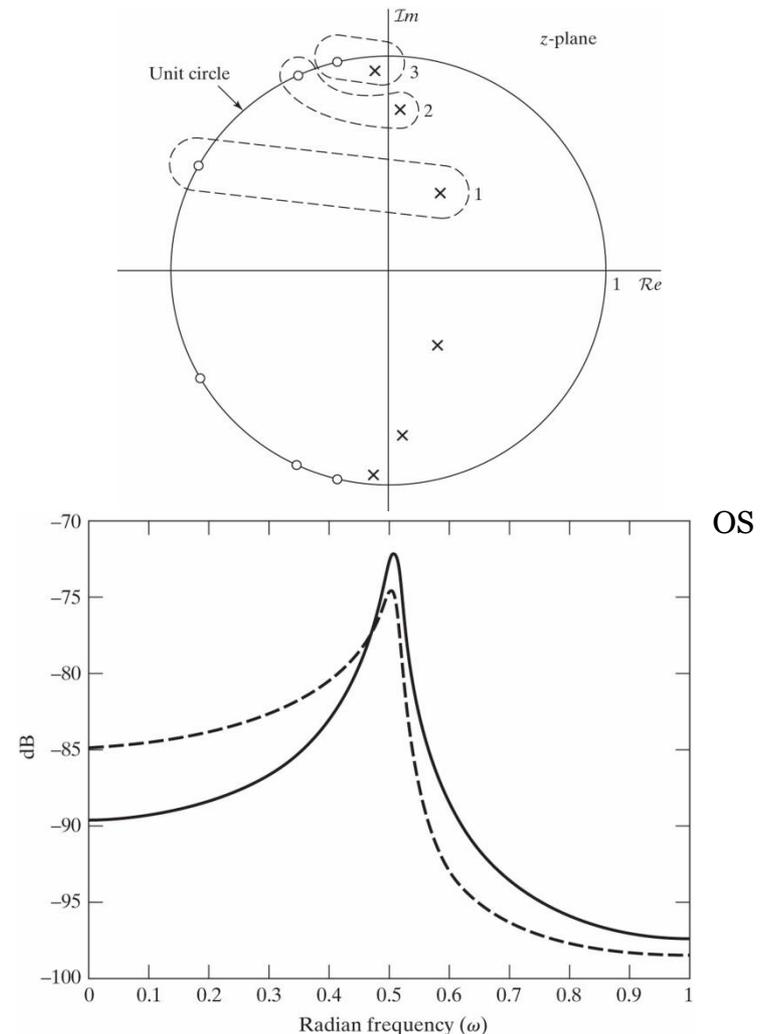


Figure 6.67 Output noise power spectrum for 123 ordering (solid line) and 321 ordering (dashed line) of 2nd-order sections.

Recursive Resonator

- Filter with frequency response dominated at a single peak
 - Use complex-conjugate pole pair inside unit circle
- $H(z) = \frac{A}{(1-r_p e^{j\omega_0} z^{-1})(1-r_p e^{-j\omega_0} z^{-1})}$
- $H(z) = \frac{A}{1-2r_p \cos(\omega_0)z^{-1}+r_p^2 z^{-2}}$
 - A – normalization constant for unity gain at ω_0
 - $0 < r_p < 1$
- Close to unit circle
 - Bandwidth $\cong 2(1 - r_p)$
 - Closer to $r_p = 1$, more peaked

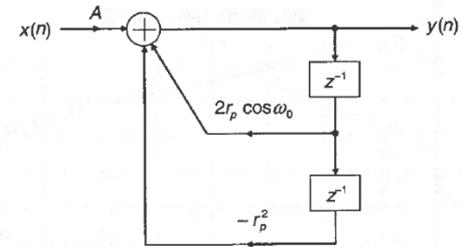
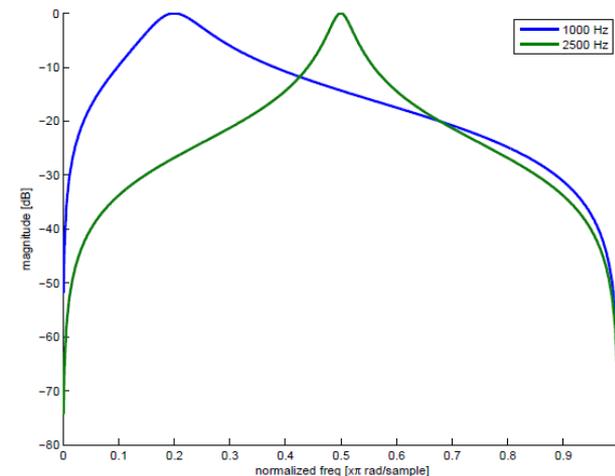
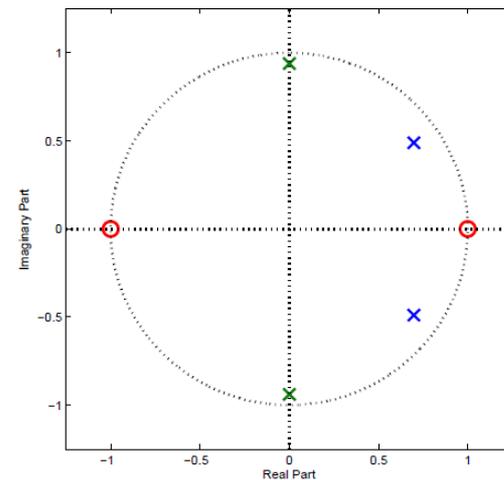


Figure 4.17 Signal-flow diagram of the second-order resonator filter



Parametric Equalizer

- Add nearby zeros to the resonator
 - At same angle as poles ω_0
 - Similar radius
- Pole and zero counter balance one another
- $r_z < r_p$
 - Pole dominates because it is closer to unit circle
 - Generates peak at $\omega = \omega_0$
 - Provides boost to freq
- $r_z > r_p$
 - Zero dominates pole
 - Generates dip at $\omega = \omega_0$
 - Cuts freq
- Bandwidth still determined by r_p

- Ex 4.18
 - Create equalizer by changing gain at given frequency

