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# EE482: Digital Signal Processing Applications

Spring 2014
TTh 14:30-15:45 CBC C222

Lecture 8
Frequency Analysis
14/02/18

http://www.ee.unlv.edu/~b1morris/ee482/

# Outline

- Fast Fourier Transform
- Butterfly Structure
- Implementation Issues

# DFT Algorithm

- The Fourier transform of an analogue signal x(t) is given by:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

- The Discrete Fourier Transform (DFT) of a discrete-time signal x(nT) is given by:

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$

  - Where:

$$k = 0, 1, \ldots N-1$$
$$x(nT) = x[n]$$


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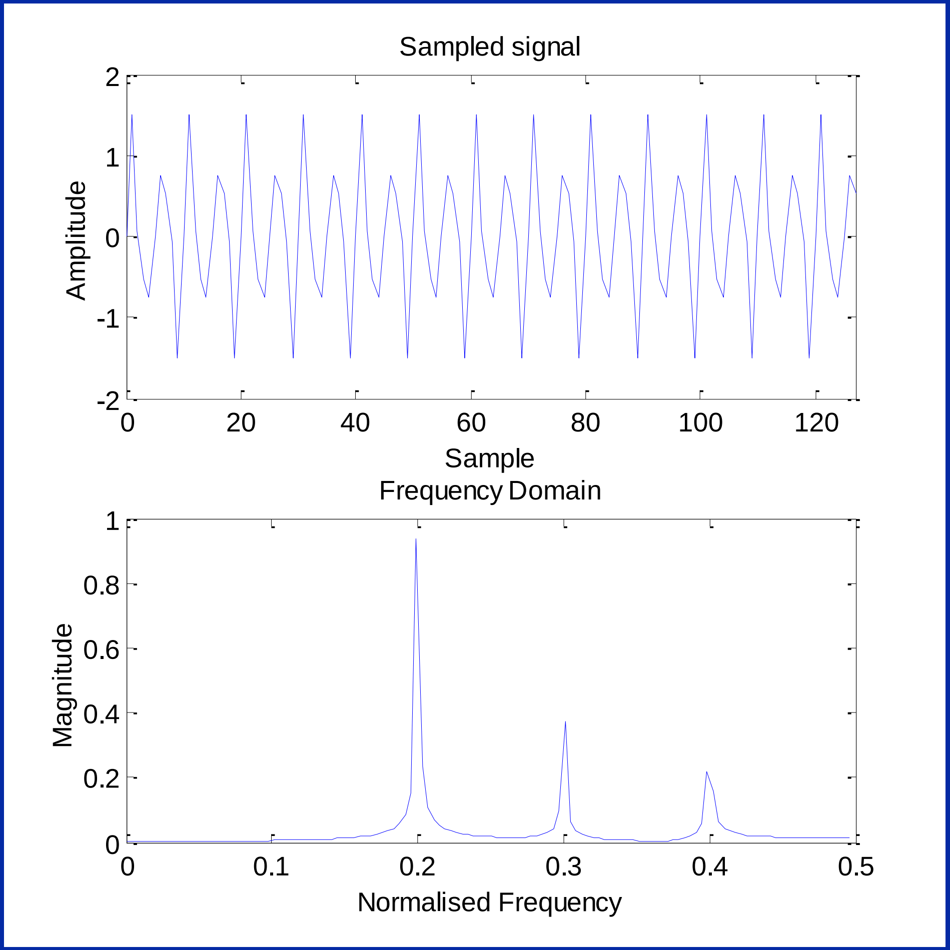
# DFT Algorithm

- If we let: $e^{-j\frac{2\pi}{N}} = W_N$ then: $X(k) = \sum_{n=0}^{N-1} x[n] W_N^{nk}$





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# DFT Algorithm

$$X(k) = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

x[n] = input

X[k] = frequency bins

W = twiddle factors

$X(0) = x[0]W_N^{0} + x[1]W_N^{0*1} + \ldots + x[N-1]W_N^{0*(N-1)}$

$X(1) = x[0]W_N^{0} + x[1]W_N^{1*1} + \ldots + x[N-1]W_N^{1*(N-1)}$

:

$X(k) = x[0]W_N^{0} + x[1]W_N^{k*1} + \ldots + x[N-1]W_N^{k*(N-1)}$

:

$X(N-1) = x[0]W_N^{0} + x[1]W_N^{(N-1)*1} + \ldots + x[N-1]W_N^{(N-1)(N-1)}$

**Note: For N samples of x we have N frequencies representing the signal.**


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# Performance of the DFT Algorithm

- The DFT requires $N^2$ (NxN) complex multiplications:
  - Each X(k) requires N complex multiplications.
  - Therefore to evaluate all the values of the DFT ( X(0) to X(N-1) ) $N^2$ multiplications are required.
- The DFT also requires (N-1)*N complex additions:
  - Each X(k) requires N-1 additions.
  - Therefore to evaluate all the values of the DFT (N-1)*N additions are required.


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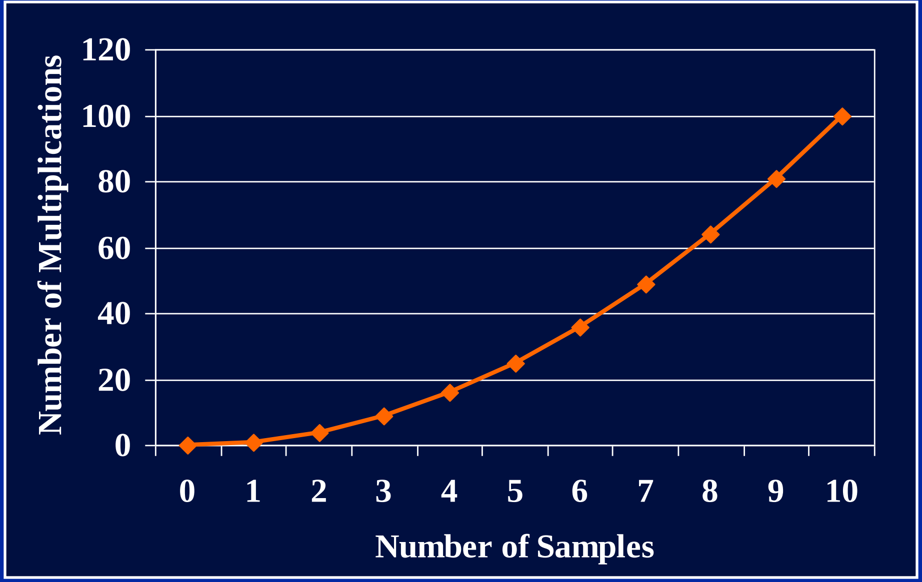
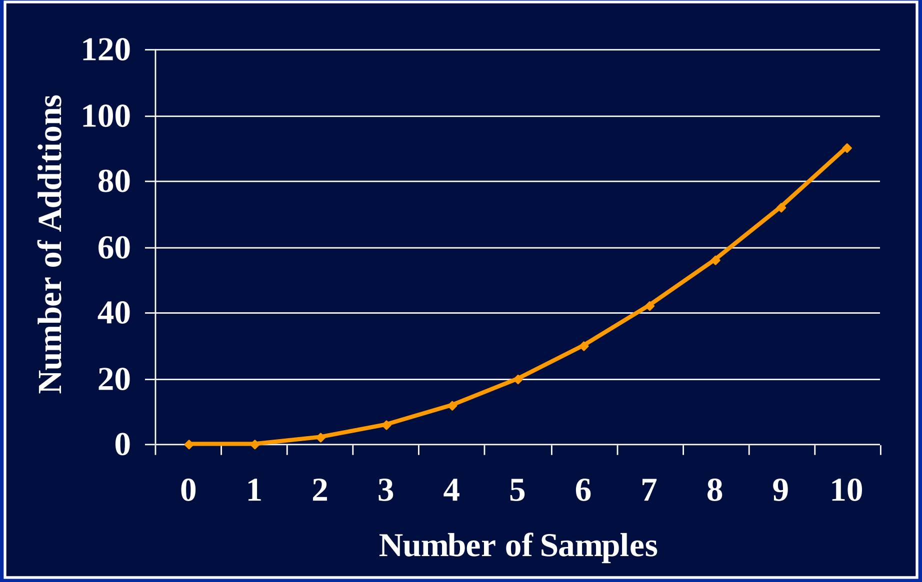
# Performance of the DFT Algorithm




- **Can the number of computations required be reduced?**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# DFT $\rightarrow$ FFT

- A large amount of work has been devoted to reducing the computation time of a DFT.
- This has led to efficient algorithms which are known as the Fast Fourier Transform (FFT) algorithms.


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# DFT → FFT

$$X(k)=\sum_{n=0}^{N-1} x[n] W_N^{nk}; \quad 0 \le k \le N-1 \qquad [1]$$

**x[n] = x[0], x[1], …, x[N-1]**

- **Lets divide the sequence x[n] into even and odd sequences:**
  - **x[2n] = x[0], x[2], …, x[N-2]**
  - **x[2n+1] = x[1], x[3], …, x[N-1]**


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# DFT → FFT

- **Equation 1 can be rewritten as:**

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1} x[2n]W_N^{2nk}+\sum_{n=0}^{\frac{N}{2}-1} x[2n+1]W_N^{(2n+1)k} \qquad [2]$$

- **Since:**

$$W_N^{2nk}=e^{-j\frac{2\pi}{N}2nk}=e^{-j\frac{2\pi}{N/2}nk}=W_{\frac{N}{2}}^{nk}$$

$$W_N^{(2n+1)k}=W_N^k\cdot W_{\frac{N}{2}}^{nk}$$

- **Then:**

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1} x[2n]W_{\frac{N}{2}}^{nk}+W_N^k\sum_{n=0}^{\frac{N}{2}-1} x[2n+1]W_{\frac{N}{2}}^{nk}=Y(k)+W_N^kZ(k)$$


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# DFT → FFT

- The result is that an N-point DFT can be divided into two N/2 point DFT's:

$$X(k)=\sum_{n=0}^{N-1} x[n]W_N^{nk};\quad 0\le k\le N-1$$

N-point DFT

- Where Y(k) and Z(k) are the two N/2 point DFTs operating on even and odd samples respectively:

$$\begin{aligned}X(k)&=\sum_{n=0}^{\frac{N}{2}-1} x_1[n]W_{\frac{N}{2}}^{nk}+W_N^k\sum_{n=0}^{\frac{N}{2}-1} x_2[n]W_{\frac{N}{2}}^{nk}\\&=Y(k)+W_N^kZ(k)\end{aligned}$$

Two N/2-point DFTs

# DFT → FFT

- **Periodicity** and **symmetry** of W can be exploited to simplify the DFT further:

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_1[n] W_{\frac{N}{2}}^{nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x_2[n] W_{\frac{N}{2}}^{nk} \quad [3]$$

$$\vdots$$

$$X\left(k+\frac{N}{2}\right) = \sum_{n=0}^{\frac{N}{2}-1} x_1[n] W_{\frac{N}{2}}^{n\left(k+\frac{N}{2}\right)} + W_N^{k+\frac{N}{2}} \sum_{n=0}^{\frac{N}{2}-1} x_2[n] W_{\frac{N}{2}}^{n\left(k+\frac{N}{2}\right)}$$

**Or:**

$$W_N^{k+\frac{N}{2}} = e^{-j\frac{2\pi}{N}k} e^{-j\frac{2\pi}{N}\frac{N}{2}} = e^{-j\frac{2\pi}{N}k} e^{-j\pi} = -e^{-j\frac{2\pi}{N}k} = -W_N^k$$

**: Symmetry**

**And:**

$$W_{\frac{N}{2}}^{k+\frac{N}{2}} = e^{-j\frac{2\pi}{N/2}k} e^{-j\frac{2\pi}{N/2}\frac{N}{2}} = e^{-j\frac{2\pi}{N/2}k} = W_{\frac{N}{2}}^k$$

**: Periodicity**


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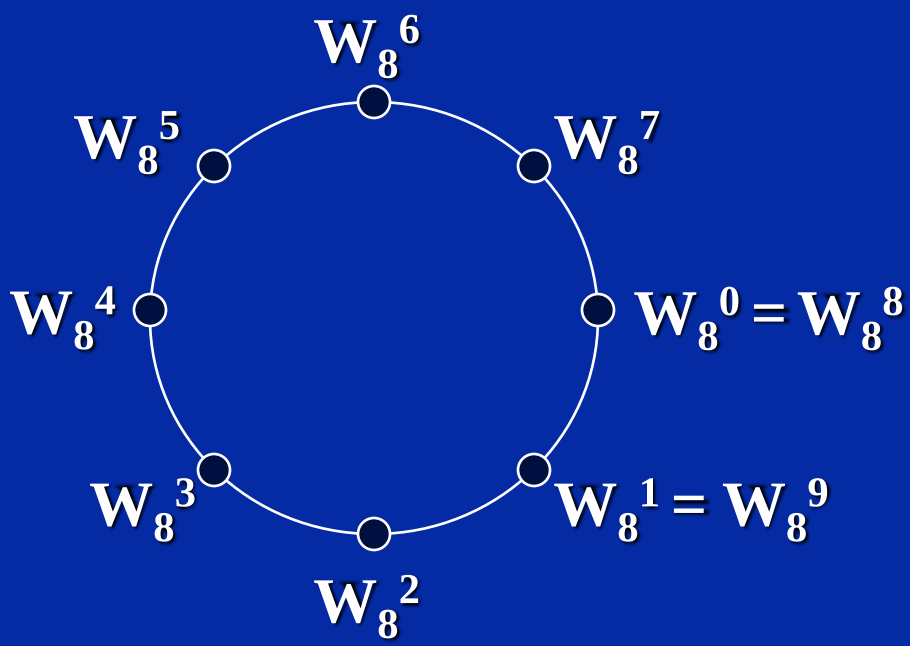
# DFT → FFT

- **Symmetry and periodicity:**




$$W_N^{k+N/2} = -W_N^k$$

$$W_{N/2}^{k+N/2} = W_{N/2}^k$$

$$W_8^{k+4} = -W_8^k$$

$$W_8^{k+8} = W_8^k$$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# DFT → FFT

- **Finally by exploiting the symmetry and periodicity, Equation 3 can be written as:**

$$X\left(k+\frac{N}{2}\right)=\sum_{n=0}^{\frac{N}{2}-1} x_1[n] W_{\frac{N}{2}}^{nk} - W_N^k \sum_{n=0}^{\frac{N}{2}-1} x_2[n] W_{\frac{N}{2}}^{nk} = Y(k)-W_N^k Z(k) \qquad [4]$$

# DFT → FFT

$$X(k) = Y(k) + W_N^k Z(k); \quad k = 0, \ldots \left(\frac{N}{2} - 1\right)$$

$$X\left(k + \frac{N}{2}\right) = Y(k) - W_N^k Z(k); \quad k = 0, \ldots \left(\frac{N}{2} - 1\right)$$

- **$Y(k)$ and $W_N^k Z(k)$ only need to be calculated once and used for both equations.**
- **Note: the calculation is reduced from 0 to N-1 to 0 to (N/2 - 1).**


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# DFT → FFT

$$X(k) = Y(k) + W_N^k Z(k); \quad k = 0, \ldots \left(\frac{N}{2} - 1\right)$$

$$X\left(k + \frac{N}{2}\right) = Y(k) - W_N^k Z(k); \quad k = 0, \ldots \left(\frac{N}{2} - 1\right)$$

- **Y(k) and Z(k) can also be divided into N/4 point DFTs using the same process shown above:**

$$Y(k) = U(k) + W_{\frac{N}{2}}^k V(k) \qquad Z(k) = P(k) + W_{\frac{N}{2}}^k Q(k)$$

$$Y\left(k + \frac{N}{4}\right) = U(k) - W_{\frac{N}{2}}^k V(k) \qquad Z\left(k + \frac{N}{4}\right) = P(k) - W_{\frac{N}{2}}^k Q(k)$$

- **The process continues until we reach 2 point DFTs.**


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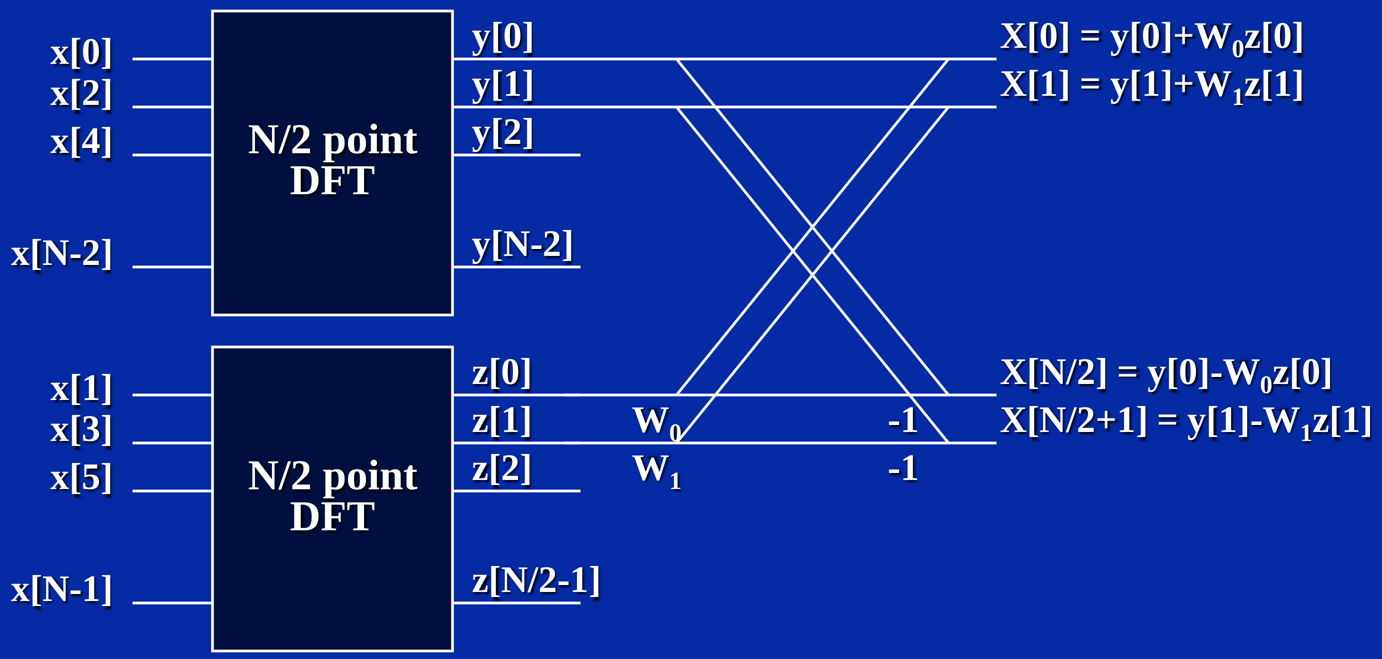
# DFT → FFT




- **Illustration of the first decimation in time FFT.**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# FFT Implementation

- To efficiently implement the FFT algorithm a few observations are made:
  - Each stage has the same number of butterflies (number of butterflies = N/2, N is number of points).
  - The number of DFT groups per stage is equal to $(N/2^{stage})$.
  - The difference between the upper and lower leg is equal to $2^{stage-1}$.
  - The number of butterflies in the group is equal to $2^{stage-1}$.


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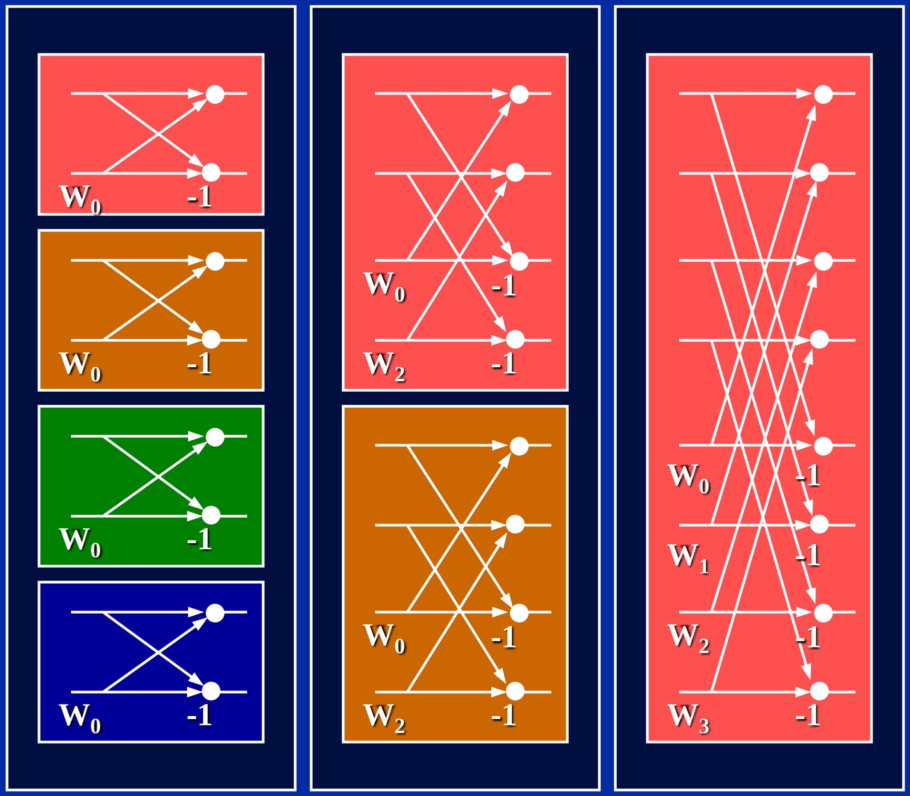
# FFT Implementation




**Example: 8 point FFT**

- **Decimation in time FFT:**
  - **Number of stages = $\log_2 N$**
  - **Number of blocks/stage = $N/2^{stage}$**
  - **Number of butterflies/block = $2^{stage-1}$**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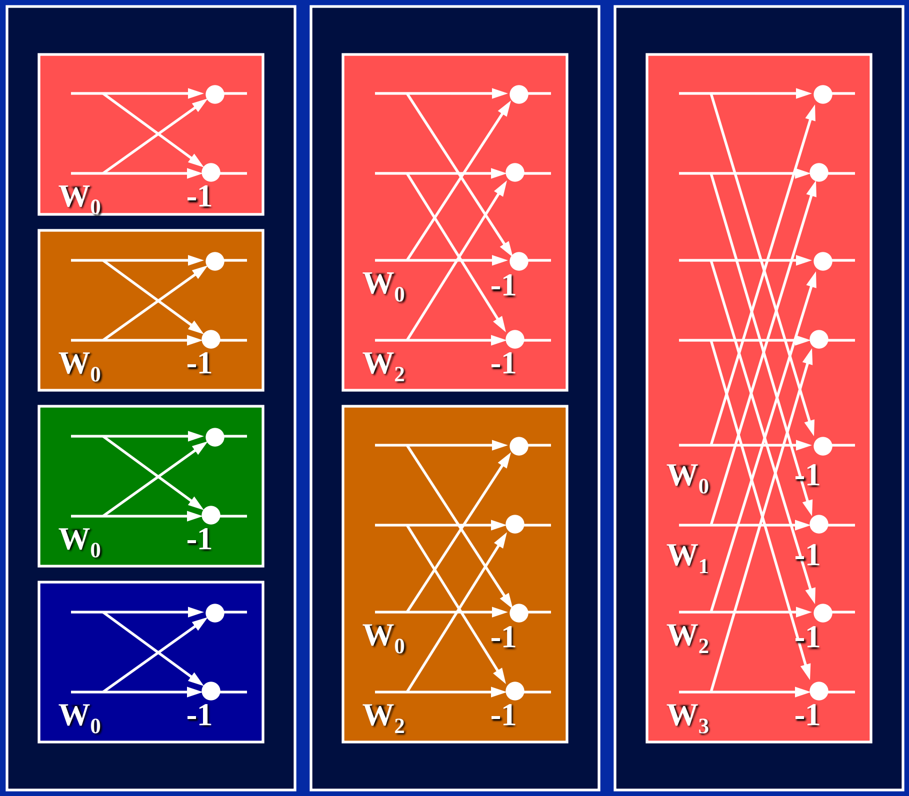
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- Decimation in time FFT:
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$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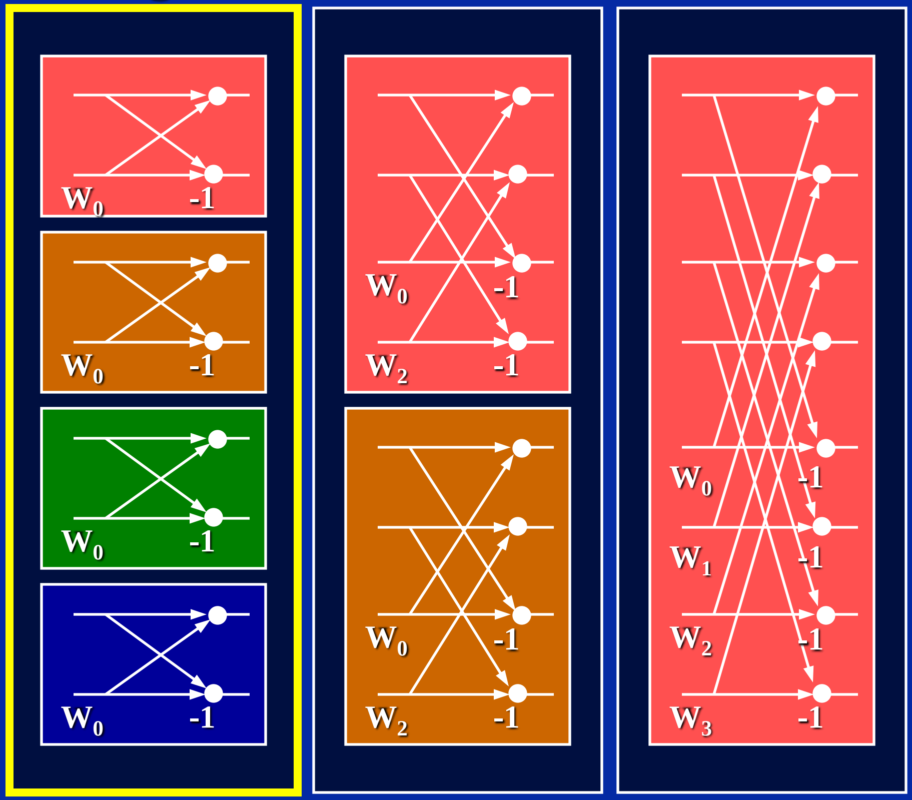
# FFT Implementation

Stage 1




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 1$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$



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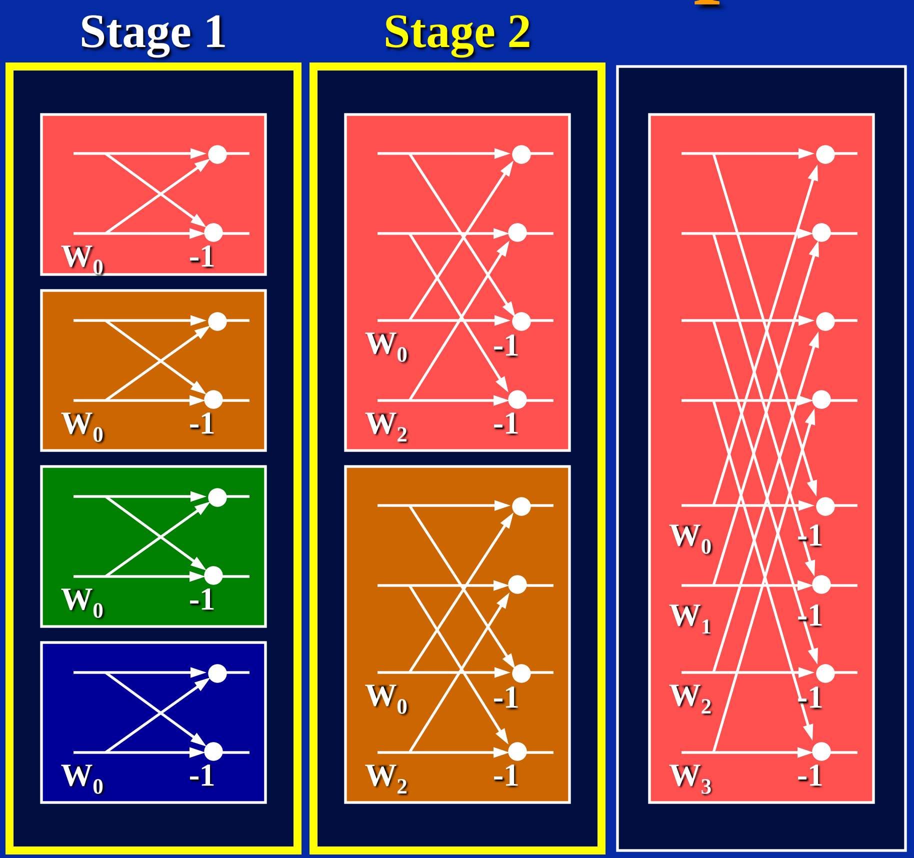
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 2$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$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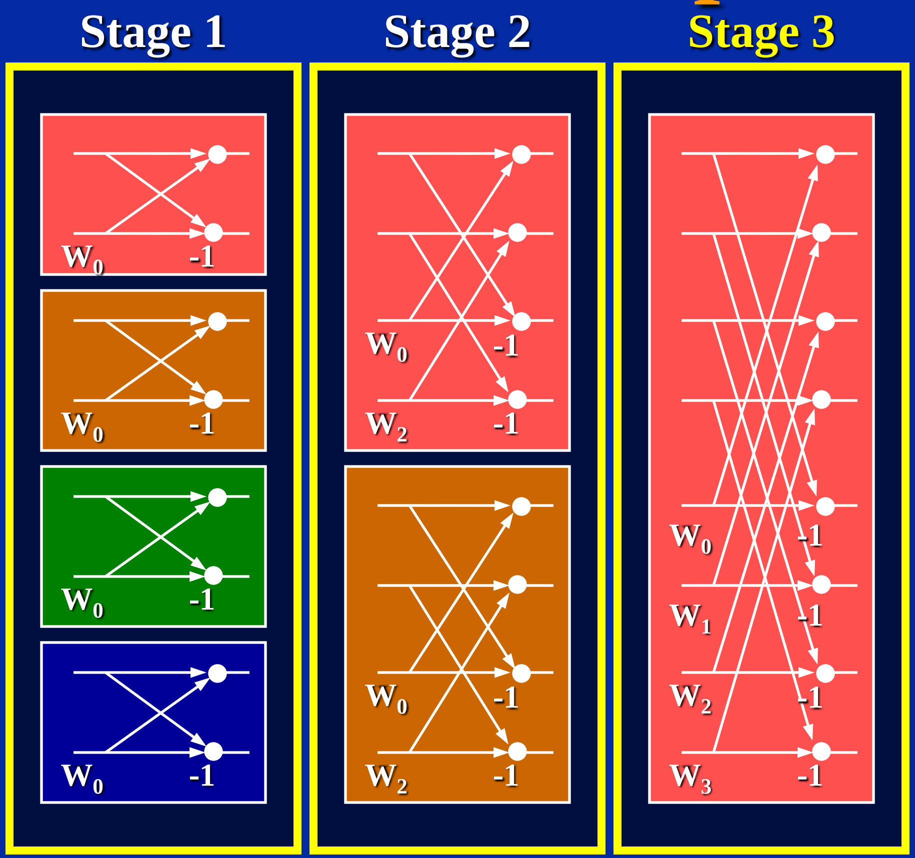
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 3$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$



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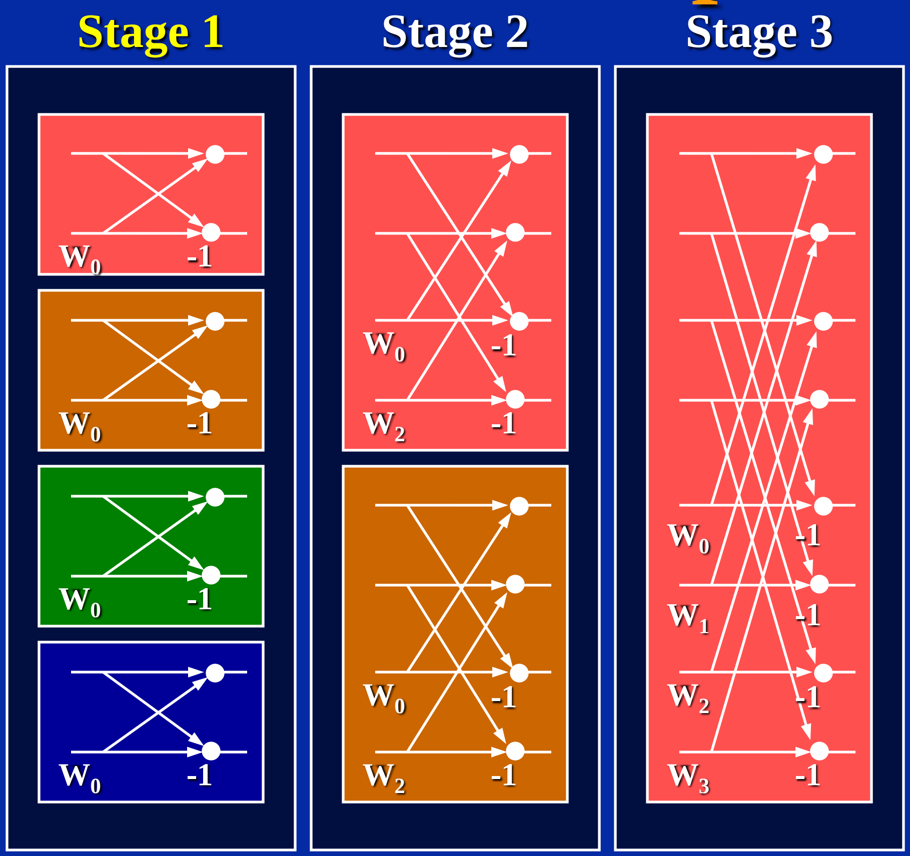
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

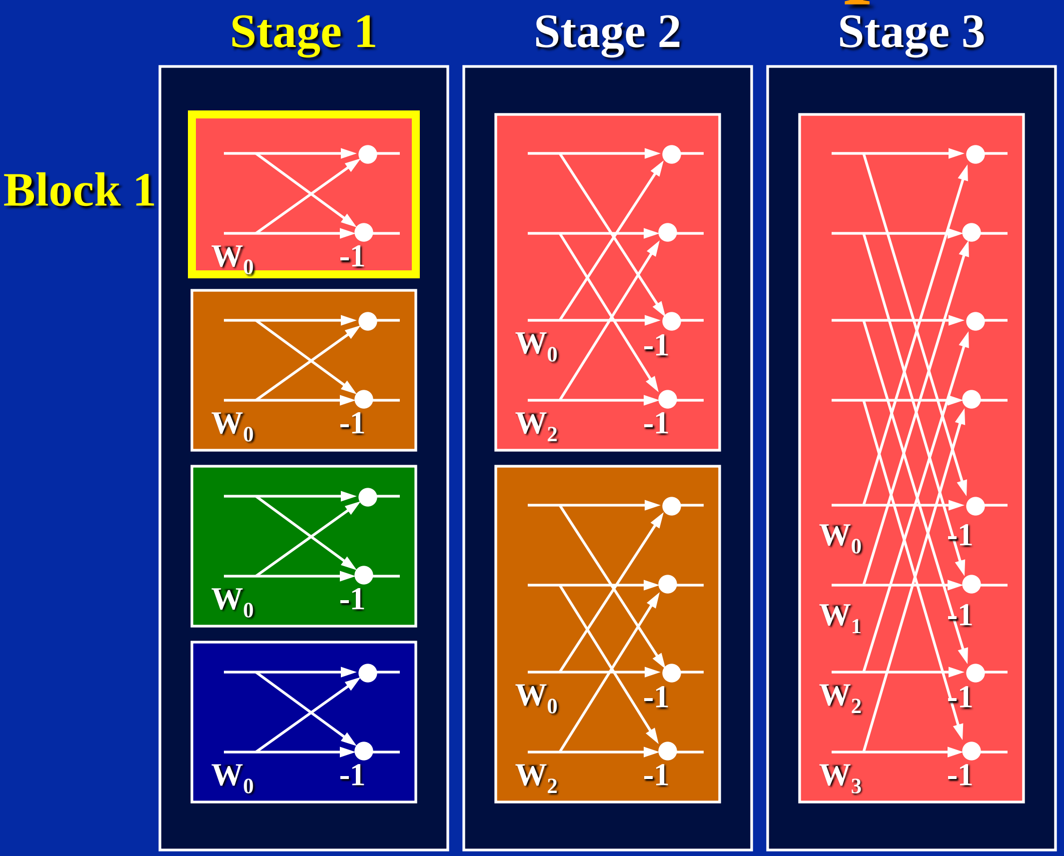
- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1:

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$

# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 1$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$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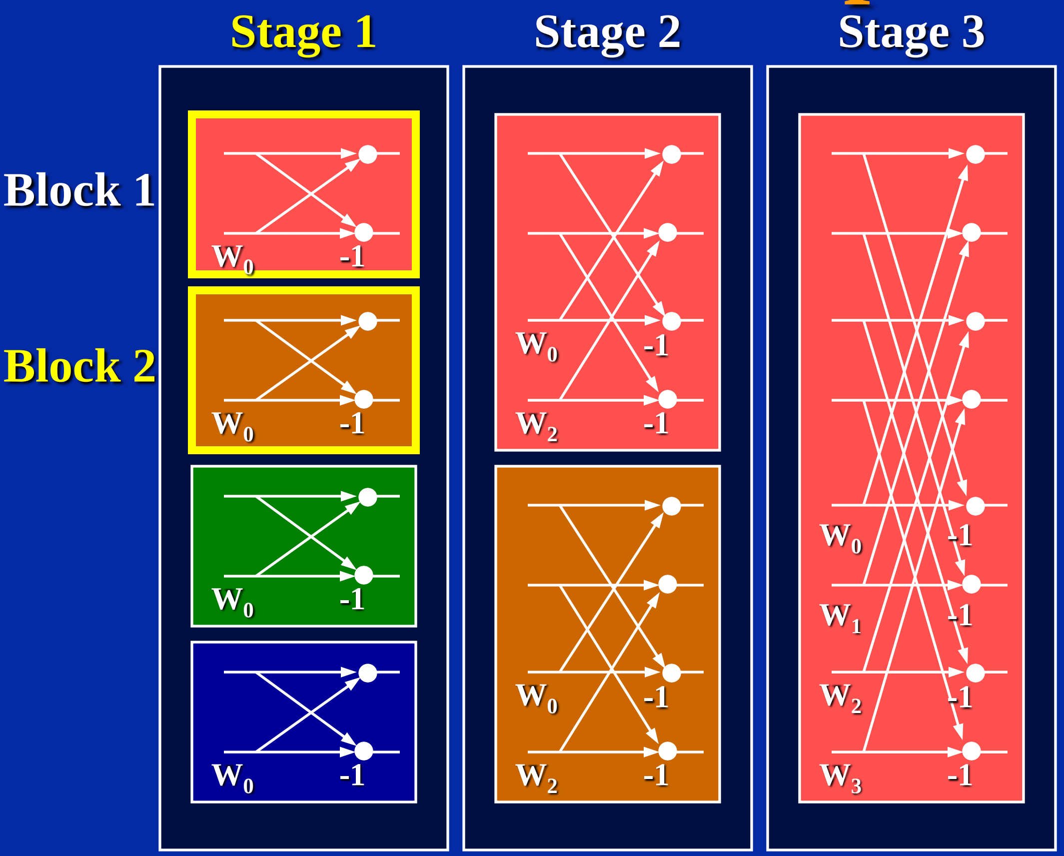
# FFT Implementation




**Example: 8 point FFT**

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 2$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$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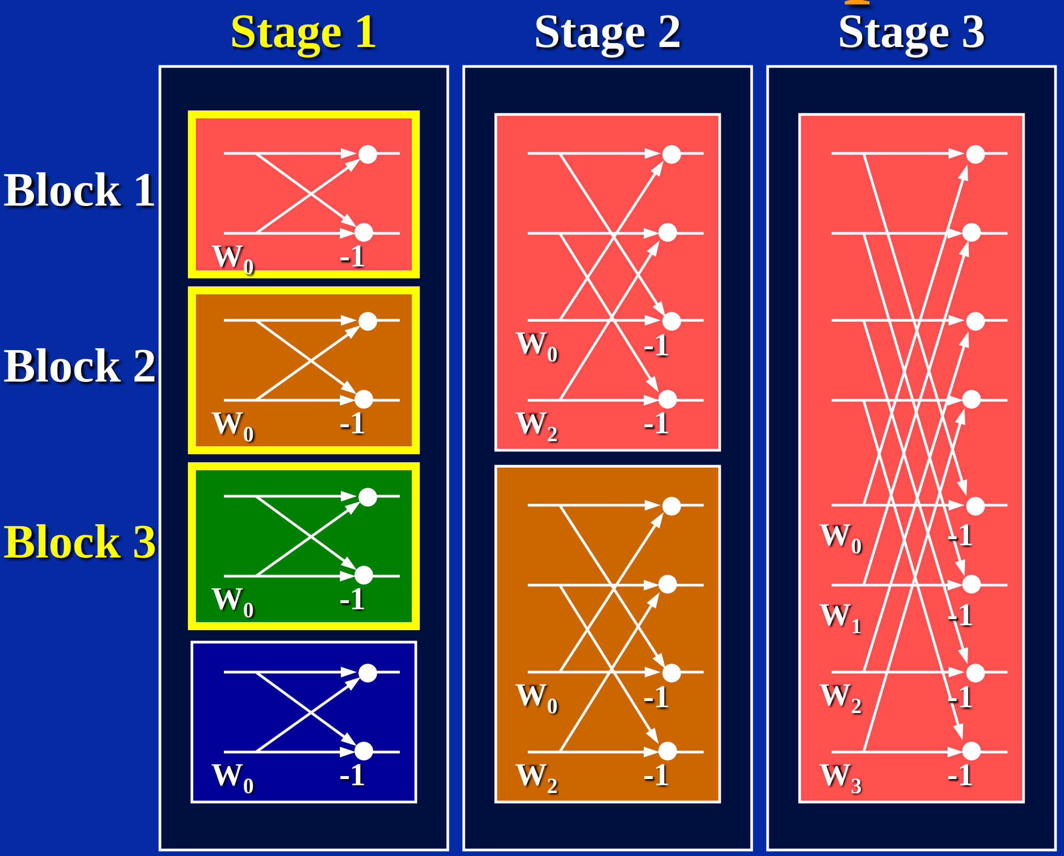
# FFT Implementation




**Example: 8 point FFT**

(1) **Number of stages:**

- $N_{stages} = 3$

(2) **Blocks/stage:**

- **Stage 1: $N_{blocks} = 3$**

- **Decimation in time FFT:**
  - **Number of stages = $\log_2 N$**
  - **Number of blocks/stage = $N/2^{stage}$**
  - **Number of butterflies/block = $2^{stage-1}$**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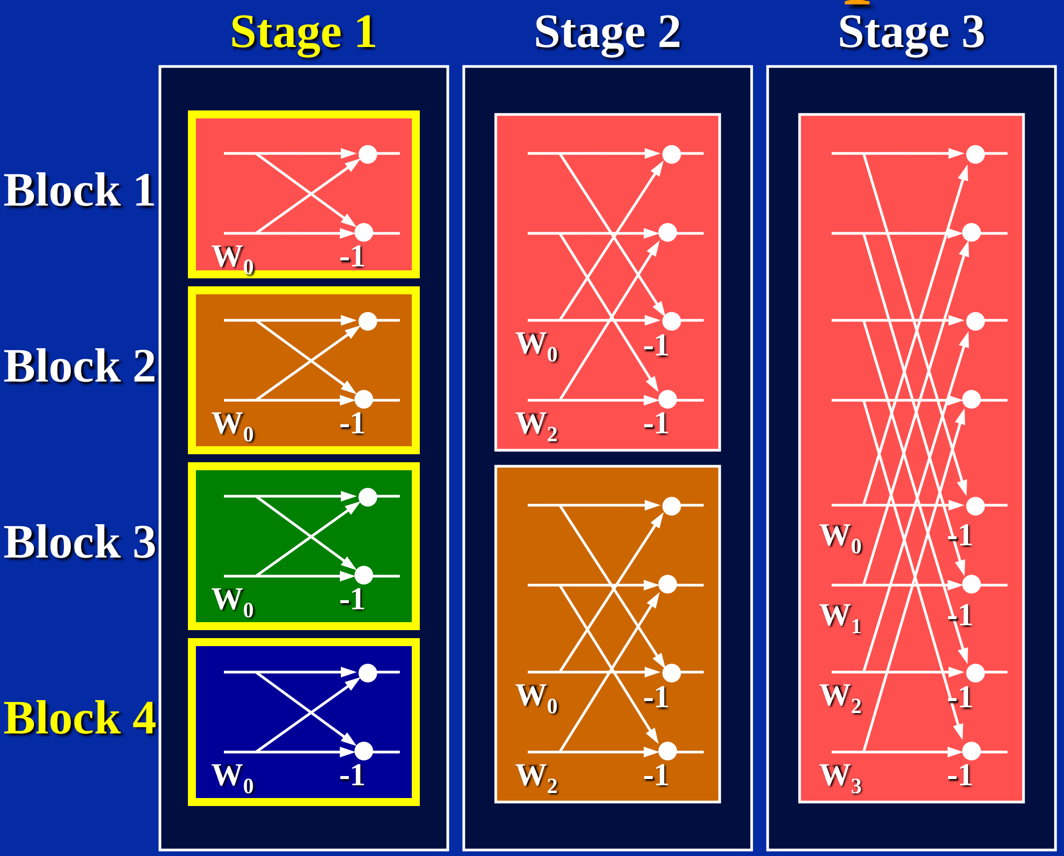
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 4$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$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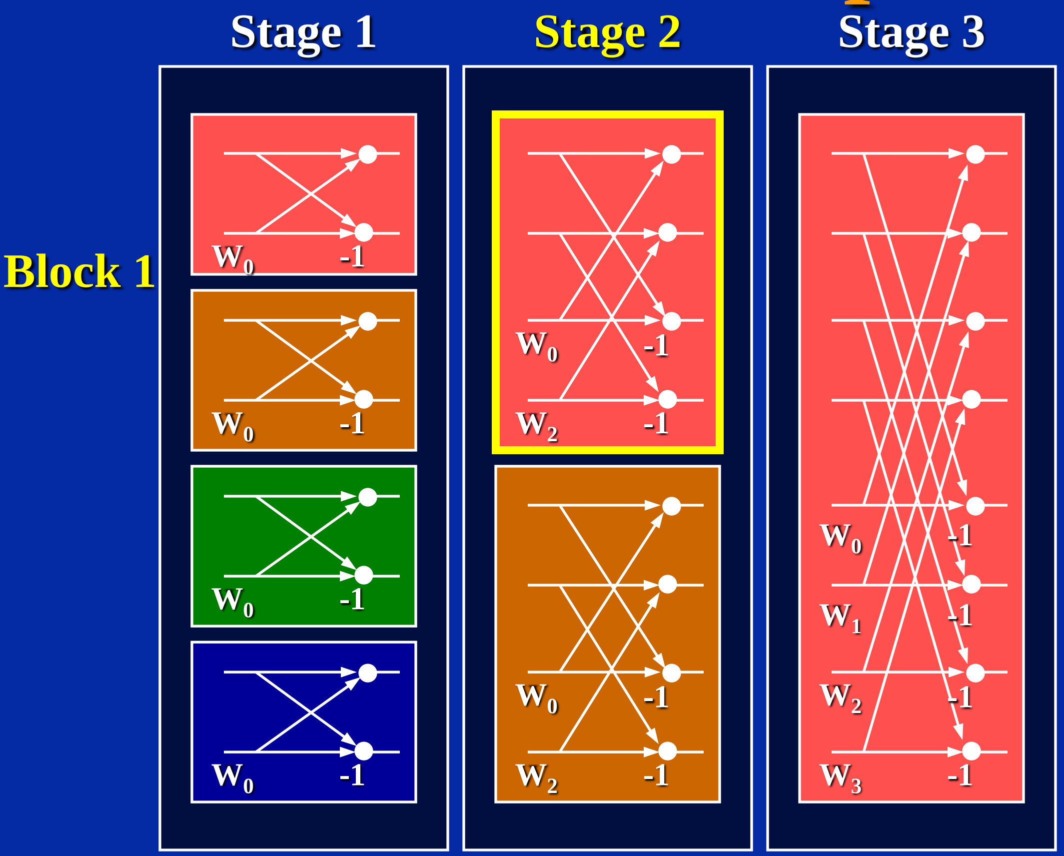
# FFT Implementation




**Example: 8 point FFT**

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 1$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$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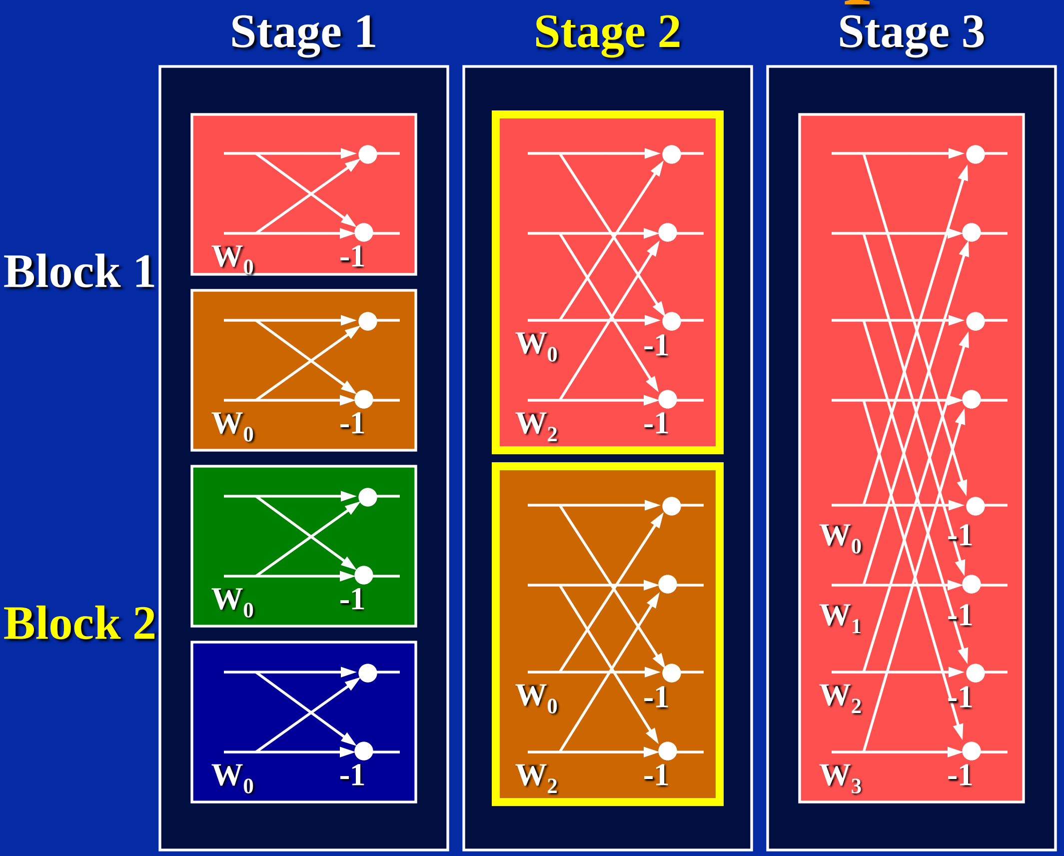
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
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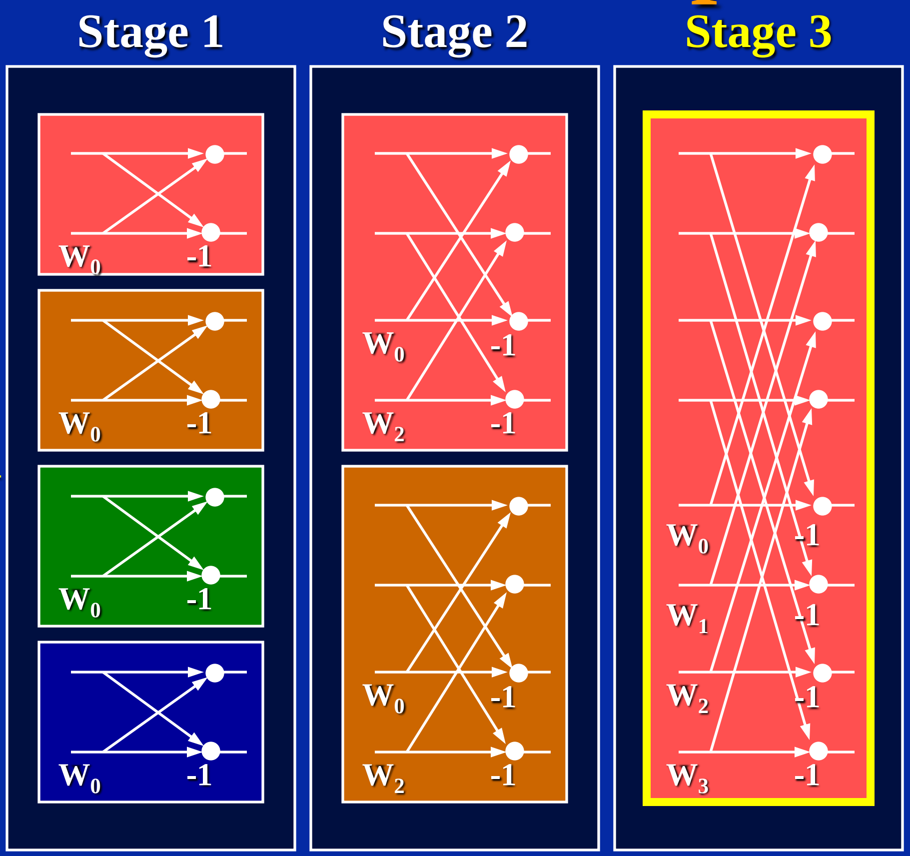
# FFT Implementation

Block 1




**Example: 8 point FFT**

**(1) Number of stages:**

- $N_{stages} = 3$

**(2) Blocks/stage:**

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$
- Stage 3: $N_{blocks} = 1$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
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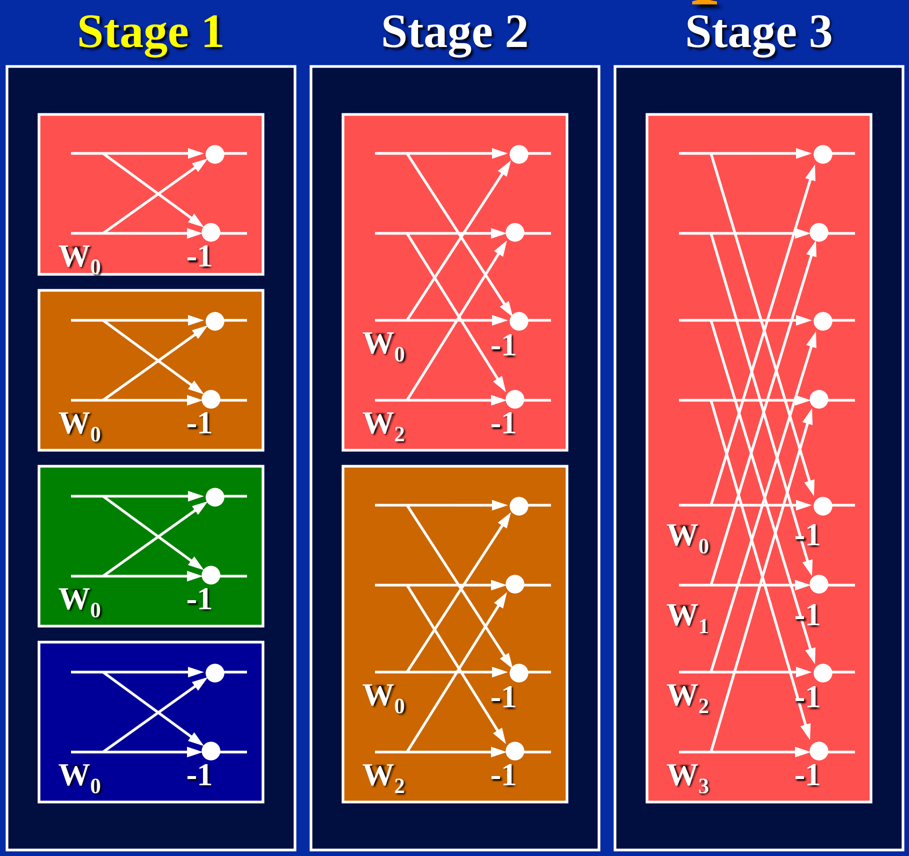
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$
- Stage 3: $N_{blocks} = 1$

(3) B'flies/block:

- Stage 1:

- Decimation in time FFT:
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
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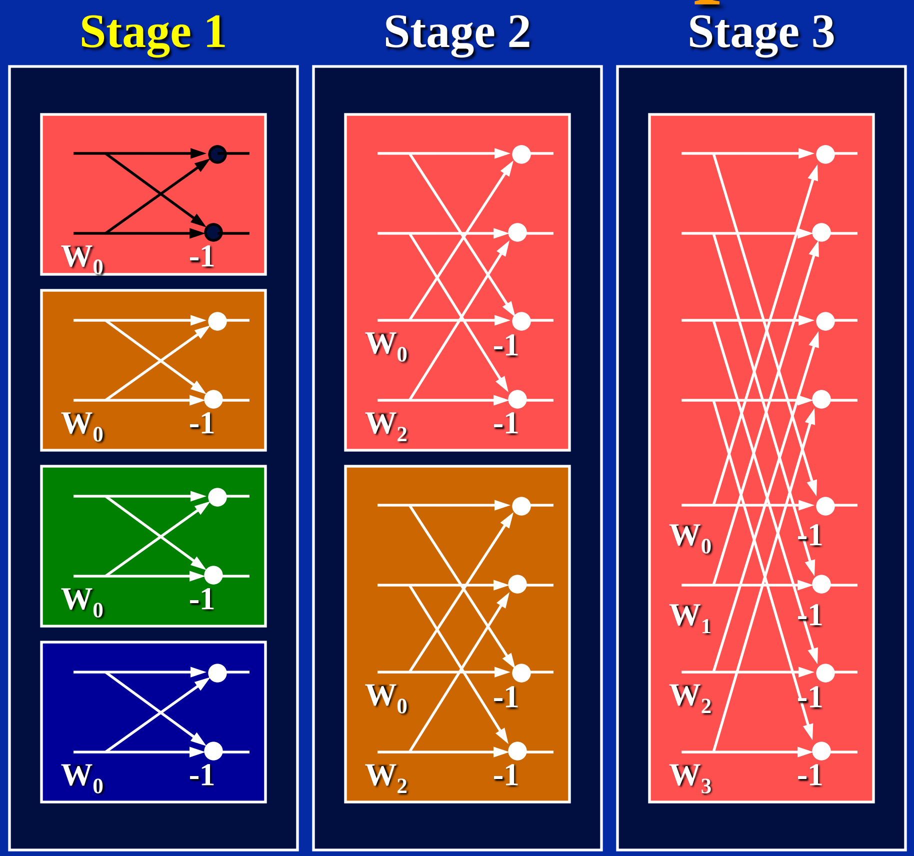
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$
- Stage 3: $N_{blocks} = 1$

(3) B'flies/block:

- Stage 1: $N_{btf} = 1$

- Decimation in time FFT:
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
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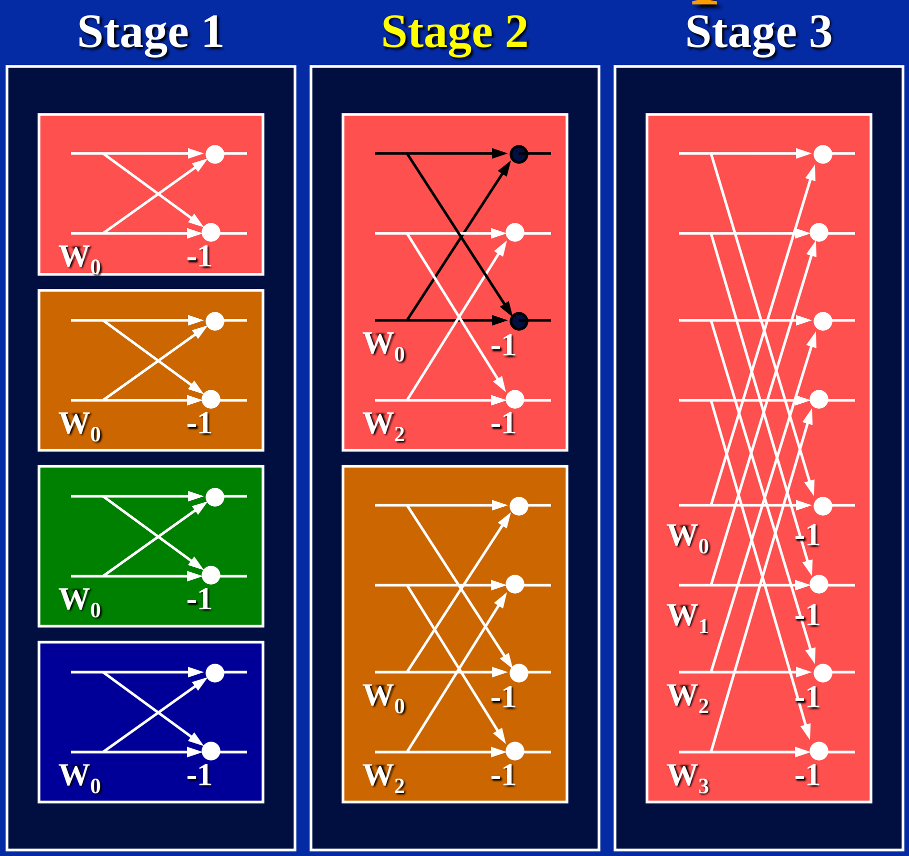
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$
- Stage 3: $N_{blocks} = 1$

(3) B'flies/block:

- Stage 1: $N_{btf} = 1$
- Stage 2: $N_{btf} = 1$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
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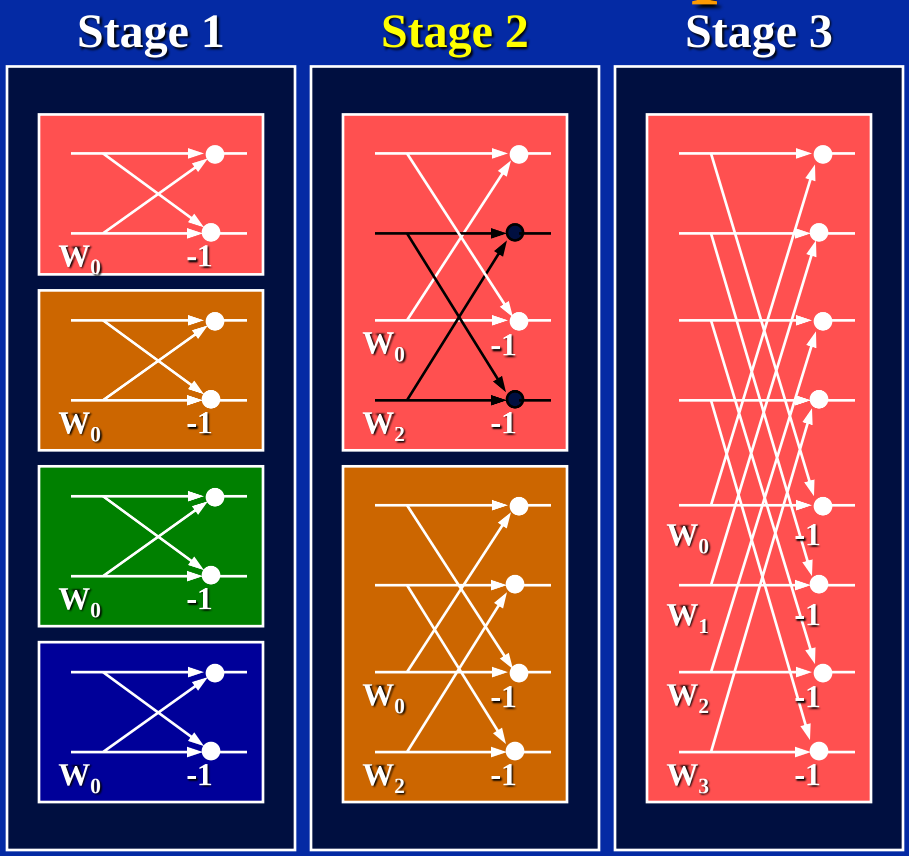
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$
- Stage 3: $N_{blocks} = 1$

(3) B'flies/block:

- Stage 1: $N_{btf} = 1$
- Stage 2: $N_{btf} = 2$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
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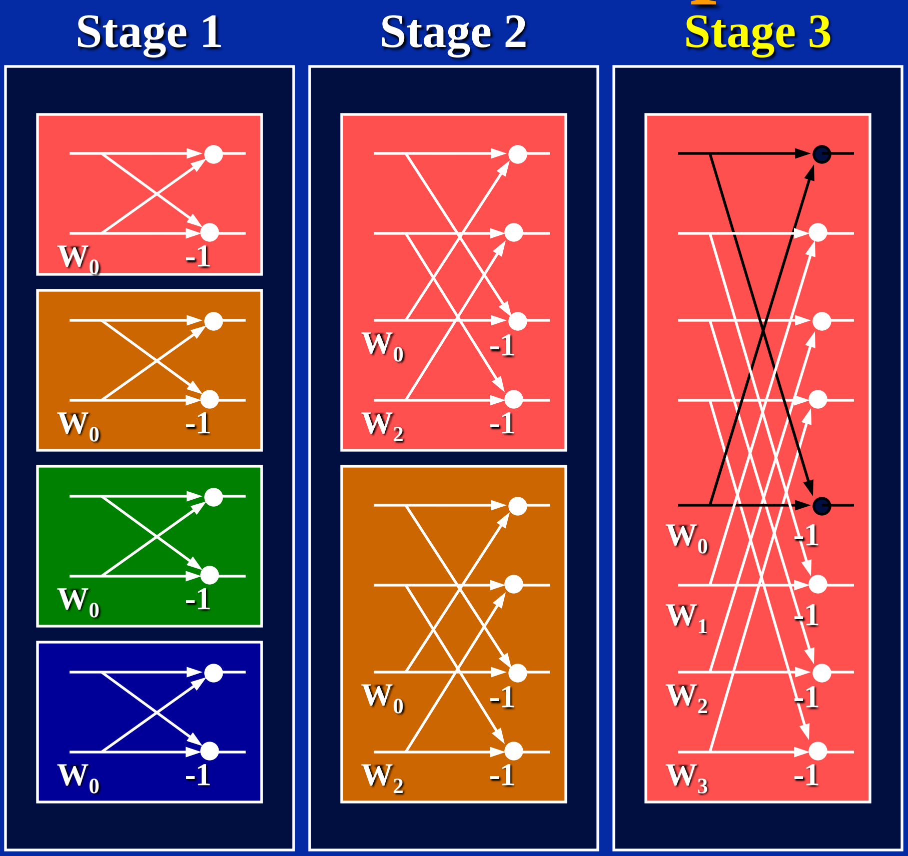
# FFT Implementation




**Example: 8 point FFT**

**(1) Number of stages:**

- $N_{stages} = 3$

**(2) Blocks/stage:**

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$
- Stage 3: $N_{blocks} = 1$

**(3) B'flies/block:**

- Stage 1: $N_{btf} = 1$
- Stage 2: $N_{btf} = 2$
- Stage 3: $N_{btf} = 1$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
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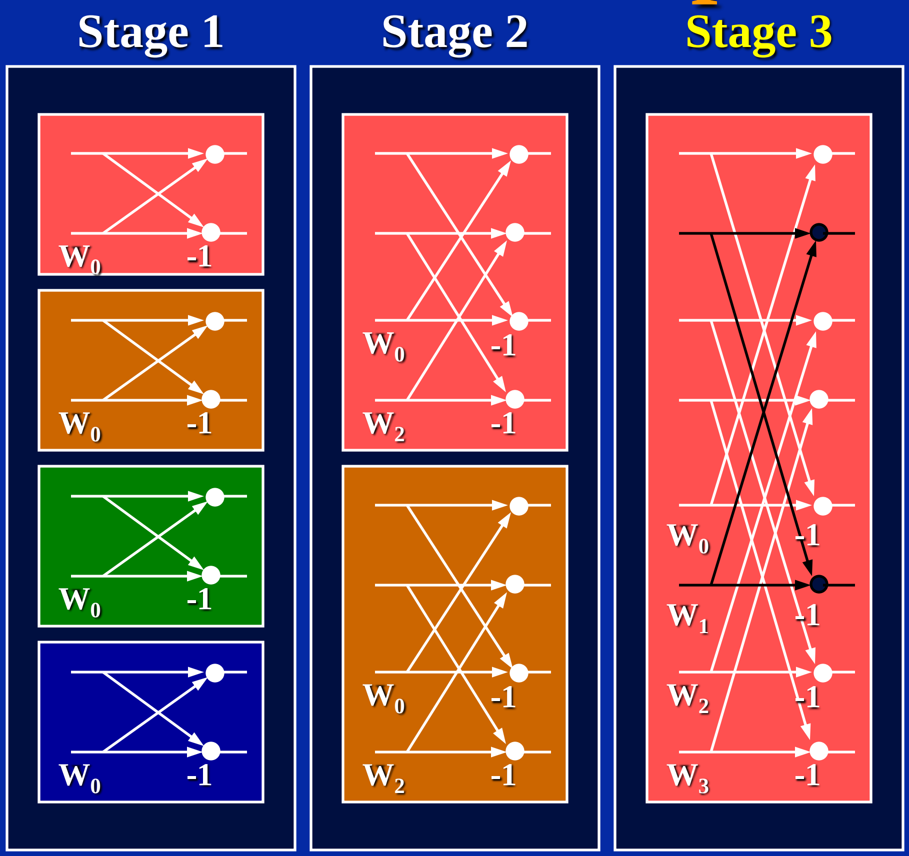
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

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(2) Blocks/stage:

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$
- Stage 3: $N_{blocks} = 1$

(3) B'flies/block:

- Stage 1: $N_{btf} = 1$
- Stage 2: $N_{btf} = 2$
- Stage 3: $N_{btf} = 2$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$


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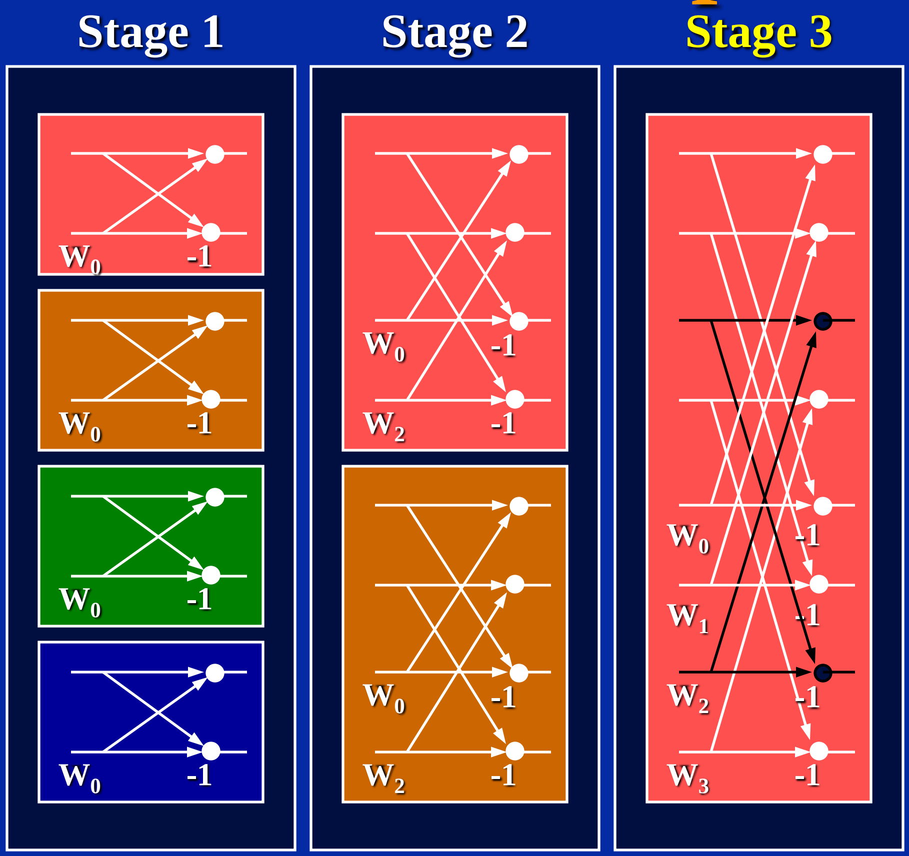
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:

- $N_{stages} = 3$

(2) Blocks/stage:

- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$
- Stage 3: $N_{blocks} = 1$

(3) B'flies/block:

- Stage 1: $N_{btf} = 1$
- Stage 2: $N_{btf} = 2$
- Stage 3: $N_{btf} = 3$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$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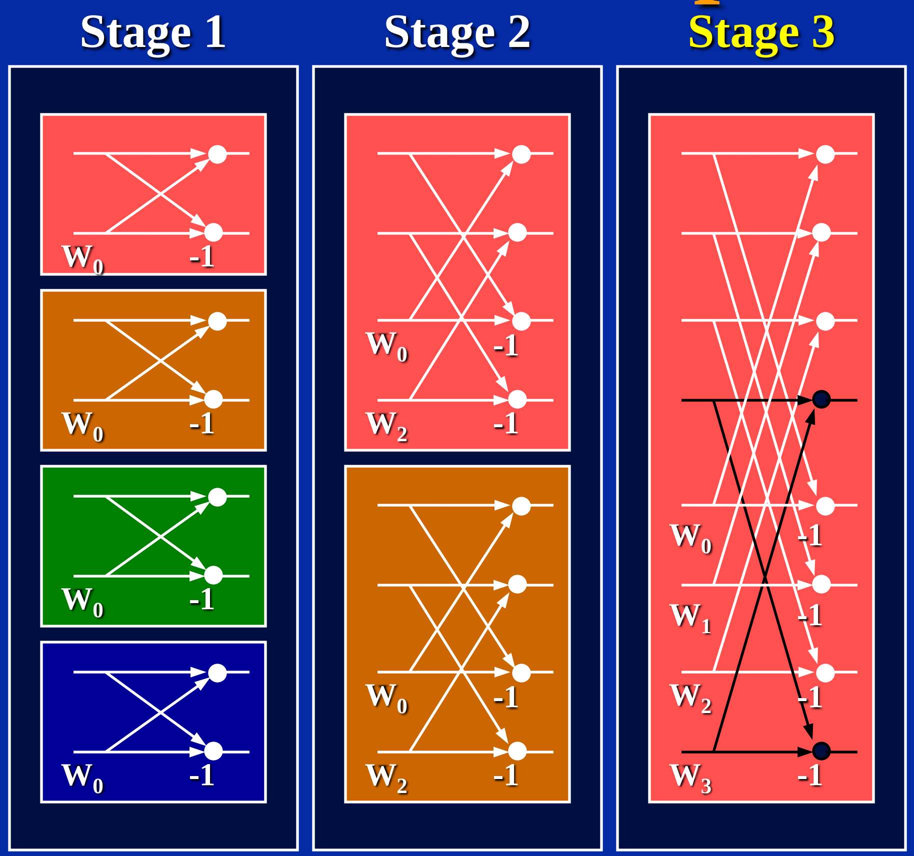
# FFT Implementation




Example: 8 point FFT

(1) Number of stages:
- $N_{stages} = 3$

(2) Blocks/stage:
- Stage 1: $N_{blocks} = 4$
- Stage 2: $N_{blocks} = 2$
- Stage 3: $N_{blocks} = 1$

(3) B'flies/block:
- Stage 1: $N_{btf} = 1$
- Stage 2: $N_{btf} = 2$
- Stage 3: $N_{btf} = 4$

- **Decimation in time FFT:**
  - Number of stages = $\log_2 N$
  - Number of blocks/stage = $N/2^{stage}$
  - Number of butterflies/block = $2^{stage-1}$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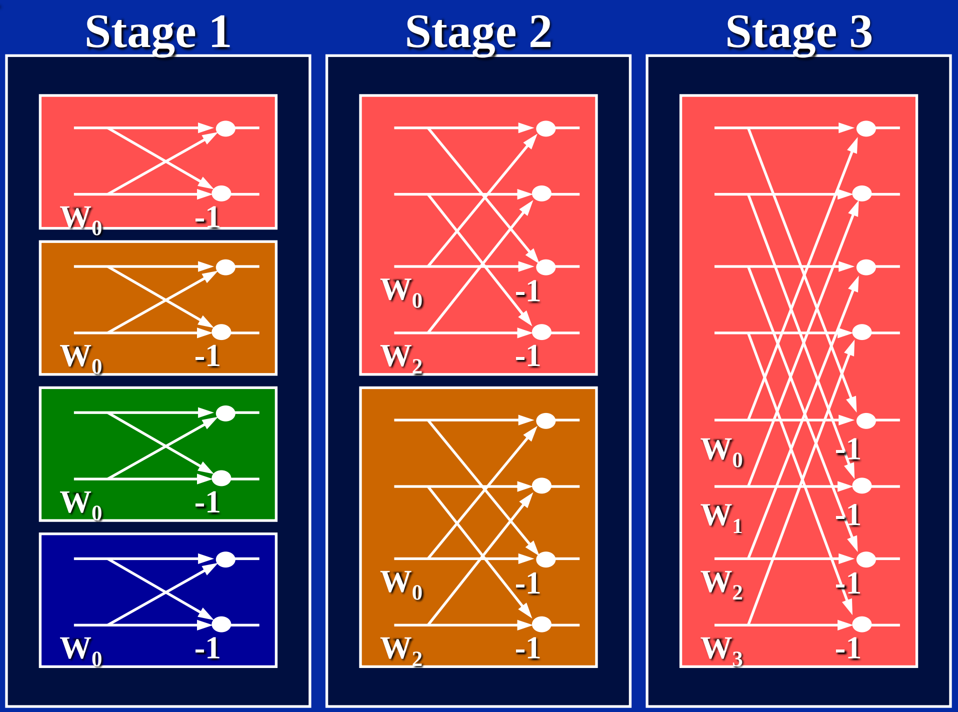
# FFT Implementation




| | Stage 1 | Stage 2 | Stage 3 |
|---|---|---|---|
| **Start Index** | **0** | **0** | **0** |
| **Input Index** | **1** | **2** | **4** |
| **Twiddle Factor Index** | **N/2 = 4** | | |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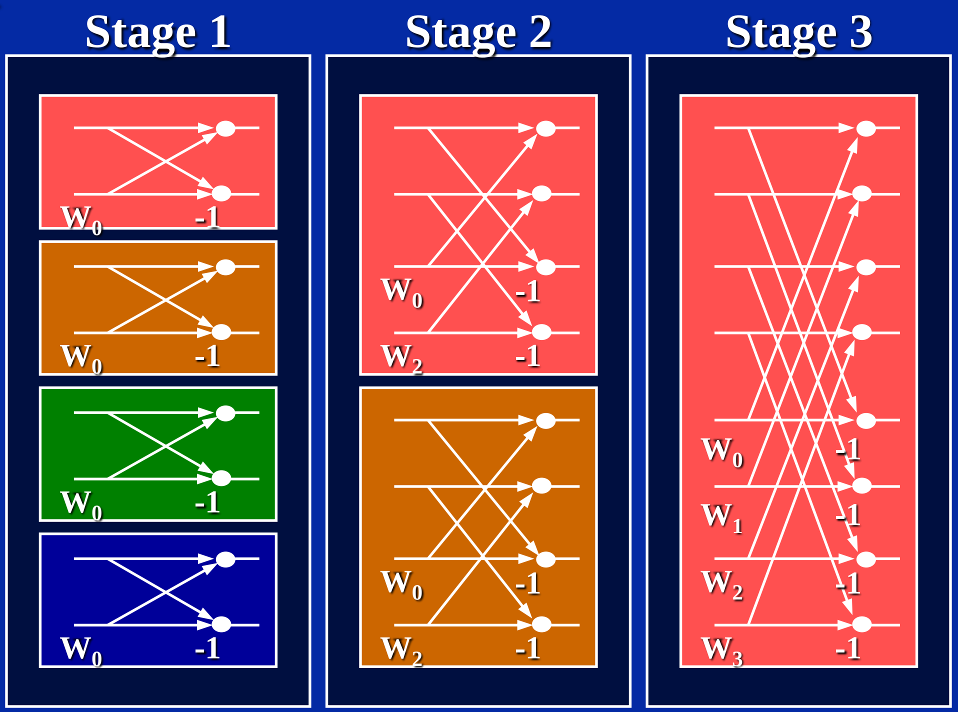
# FFT Implementation




| | Stage 1 | Stage 2 | Stage 3 |
|---|---|---|---|
| **Start Index** | 0 | 0 | 0 |
| **Input Index** | 1 | 2 | 4 |
| **Twiddle Factor Index** | N/2 = 4 | 4/2 = 2 | |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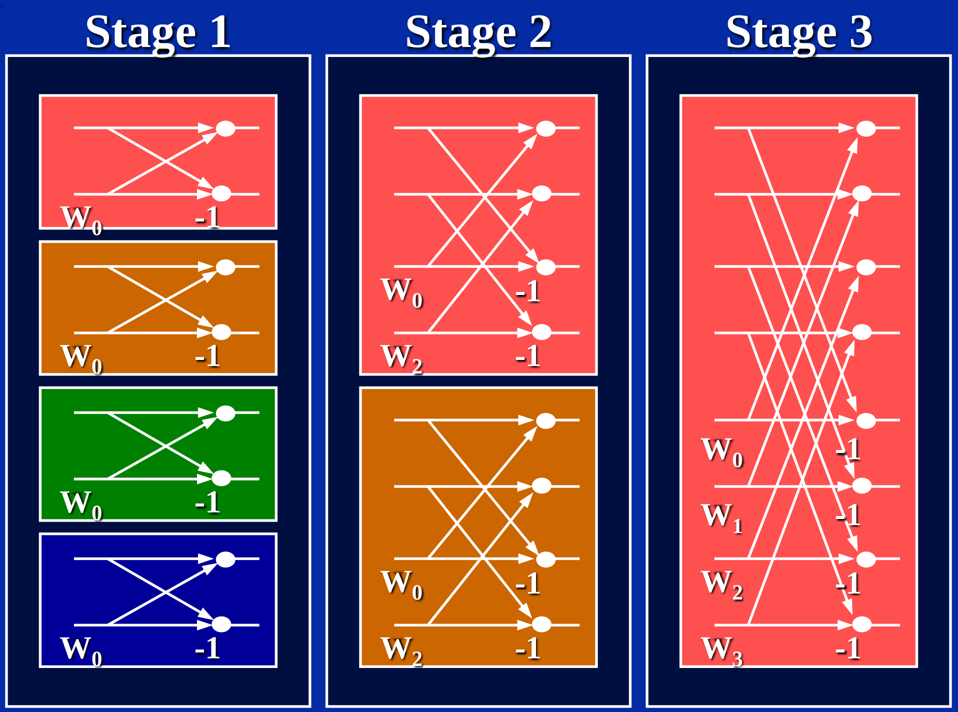
# FFT Implementation




| | Stage 1 | Stage 2 | Stage 3 |
|---|---|---|---|
| **Start Index** | 0 | 0 | 0 |
| **Input Index** | 1 | 2 | 4 |
| **Twiddle Factor Index** | N/2 = 4 | 4/2 = 2 | 2/2 = 1 |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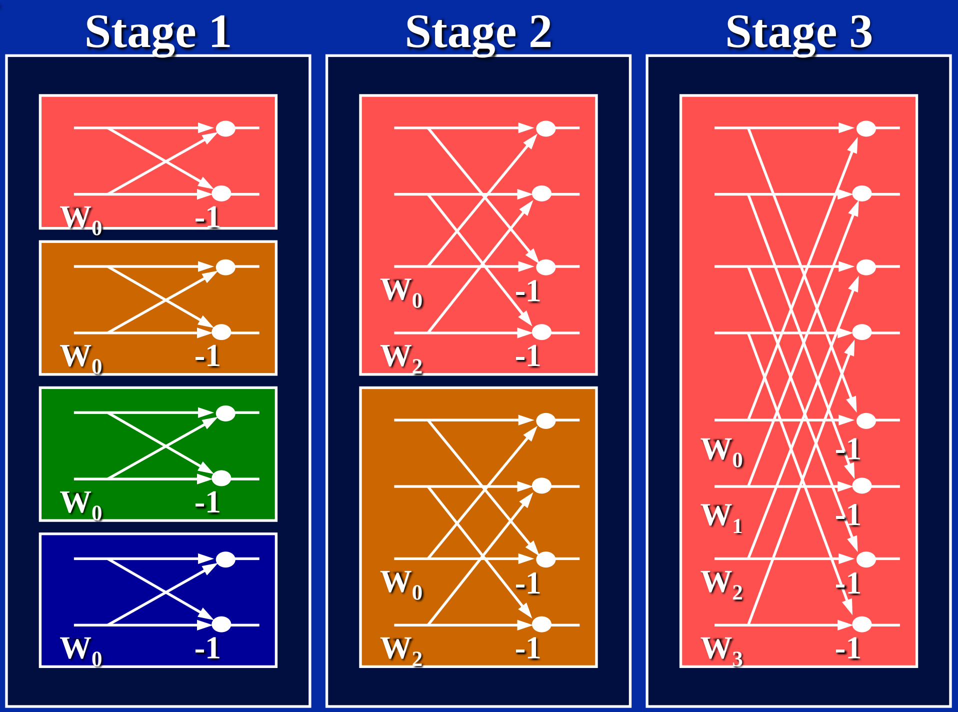
# FFT Implementation




| | Stage 1 | Stage 2 | Stage 3 |
|---|---|---|---|
| **Start Index** | 0 | 0 | 0 |
| **Input Index** | 1 | 2 | 4 |
| **Twiddle Factor Index** | N/2 = 4 | 4/2 = 2 | 2/2 = 1 |
| **Indicies Used** | $W_0$ | $W_0$, $W_2$ | $W_0$, $W_1$, $W_2$, $W_3$ |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# FFT Decimation in Frequency

- Similar divide and conquer strategy
  - Decimate in frequency domain
- $X(2k) = \sum_{n=0}^{N-1} x(n) W_N^{2nk}$
- $X(2k) = \sum_{n=0}^{N/2-1} x(n) W_{N/2}^{nk} + \sum_{n=N/2}^{N-1} x(n) W_{N/2}^{nk}$
  - Divide into first half and second half of sequence
- $X(2k) =$
  $\sum_{n=0}^{N/2-1} x(n) W_{N/2}^{nk} + \sum_{n=0}^{N/2-1} x\left(n + \frac{N}{2}\right) W_{N/2}^{\left(n+\frac{N}{2}\right)k}$
- Simplifying with twidle properties
  - $X(2k) = \sum_{n=0}^{N/2-1} \left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_{N/2}^{nk}$
  - $X(2k+1) = \sum_{n=0}^{N/2-1} W_N^n \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_{N/2}^{nk}$

# FFT Decimation in Frequency Structure

- Stage structure

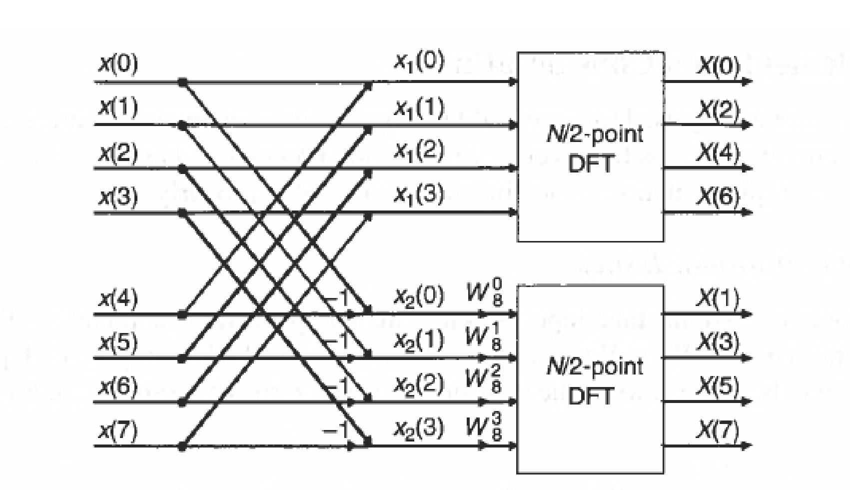



**Figure 5.8** Decomposition of an *N*-point DFT into two *N*/2-point DFTs

- Full structure

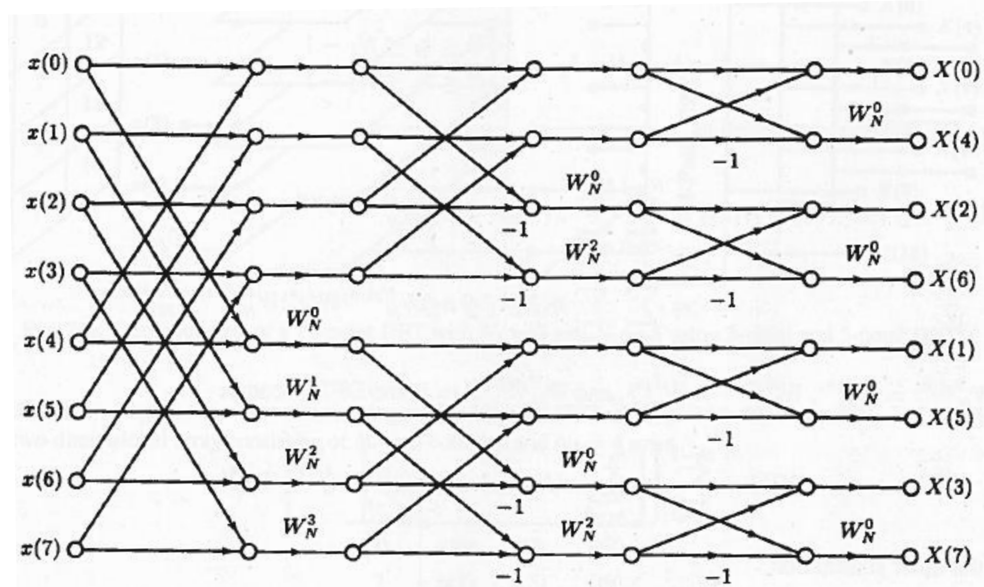



**Fig. 7-8.** Eight-point radix-2 decimation-in-frequency FFT.

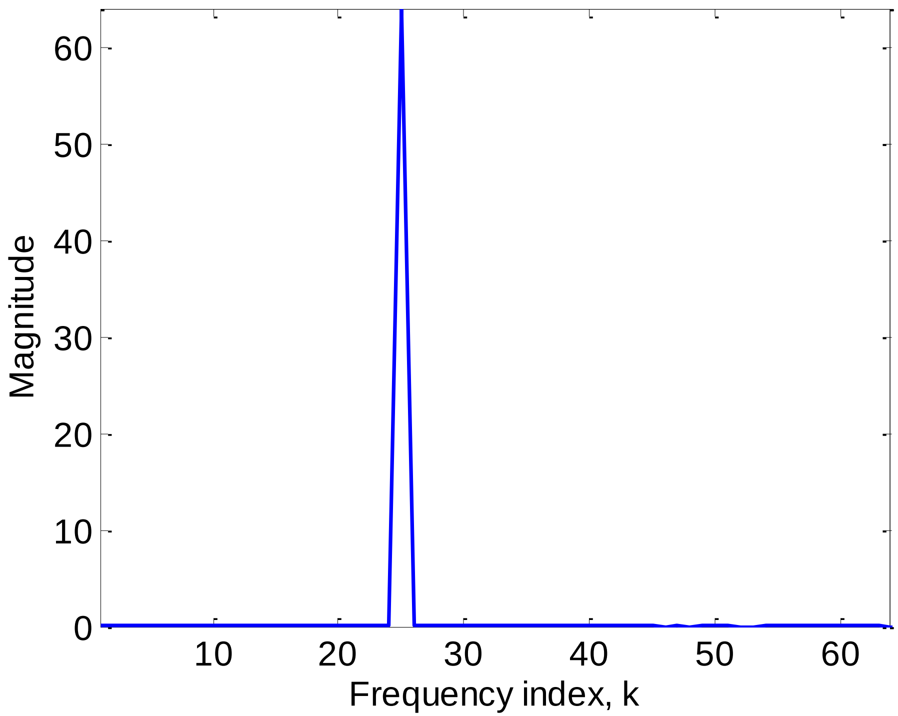
- Bit reversal happens at output instead of input

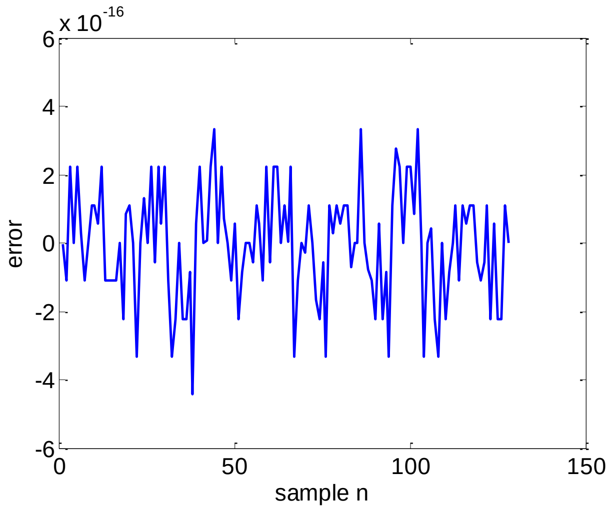
# Inverse FFT

- $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$
- Notice this is the DFT with a scale factor and change in twidle sign
- Can compute using the FFT with minor modifications
  - $x^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, W_N^{kn}$
    - Conjugate coefficients, compute FFT with scale factor, conjugate result
    - For real signals, no final conjugate needed
  - Can complex conjugate twidle factors and use in butterfly structure

# FFT Example

- Example 5.10
- Sine wave with $f = 50$ Hz
  - $x(n) = \sin\left(\frac{2\pi f n}{f_s}\right)$
    - $n = 0,1,\ldots,128$
    - $f_s = 256$ Hz
- Frequency resolution of DFT?
  - $\Delta = f_s/N = \frac{256}{128} = 2$ Hz
- Location of peak
  - $50 = k\Delta \rightarrow k = \frac{50}{2} = 25$

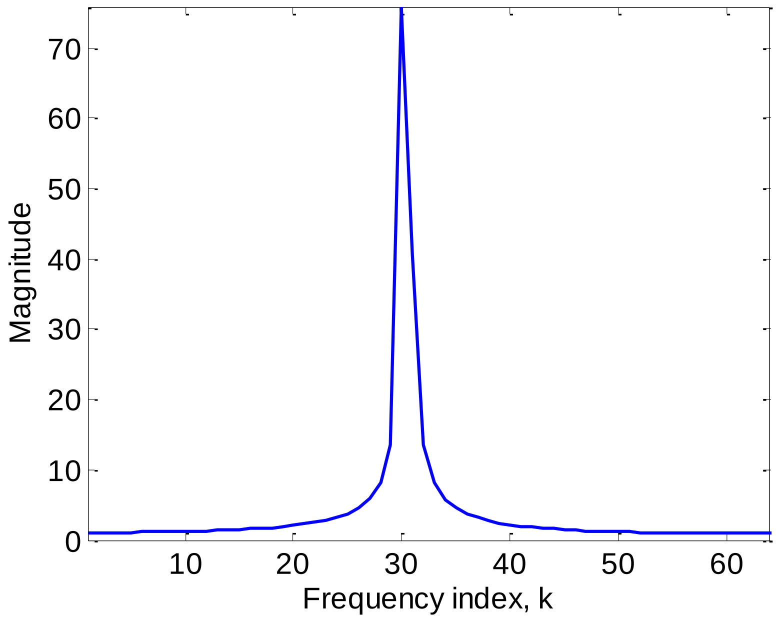


# Spectral Leakage and Resolution

- Notice that a DFT is like windowing a signal to finite length
  - Longer window lengths (more samples) the closer DFT $X(k)$ approximates DTFT $X(\omega)$
- Convolution relationship
  - $x_N(n) = w(n)x(n)$
  - $X_N(k) = W(k) * X(k)$
- Corruption of spectrum due to window properties (mainlobe/sidelobe)
  - Sidelobes result in spurious peaks in computed spectrum known as spectral leakage
    - Obviously, want to use smoother windows to minimize these effects
  - Spectral smearing is the loss in sharpness due to convolution which depends on mainlobe width

- Example 5.15
  - Two close sinusoids smeared together

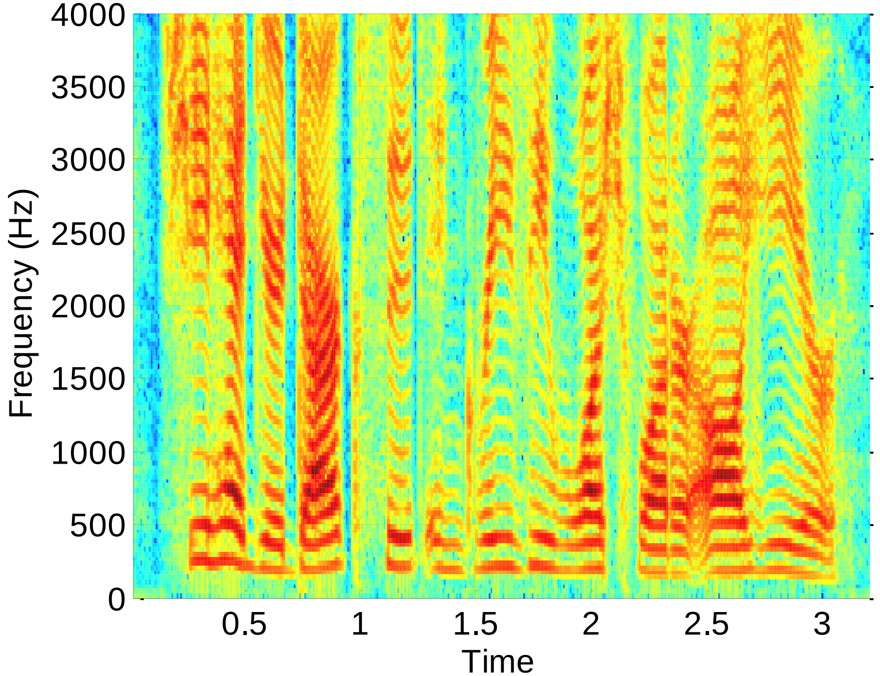



- To avoid smearing:
  - Frequency separation should be greater than freq resolution
  - $N > \frac{2\pi}{\Delta\omega}, \quad N > f_s/\Delta f$

# Power Spectral Density

- Parseval's theorem
- $E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$
  - $|X(k)|^2$ - power spectrum or periodogram
- Power spectral density (PSD, or power density spectrum or power spectrum) is used to measure average power over frequencies
- Computed for time-varying signal by using a sliding window technique
  - Short-time Fourier transform
  - Grab $N$ samples and compute FFT
    - Must have overlap and use windows

- Spectrogram
  - Each short FFT is arranged as a column in a matrix to give the time-varying properties of the signal
  - Viewed as an image




"She had your dark suit in greasy wash water all year"

# Fast FFT Convolution

- Linear convolution is multiplication in frequency domain
  - Must take FFT of signal and filter, multiply, and iFFT
  - Operations in frequency domain can be much faster for large filters
  - Requires zero-padding because of circular convolution
- Typically, will do block processing
  - Segment a signal and process each segment individually before recombining

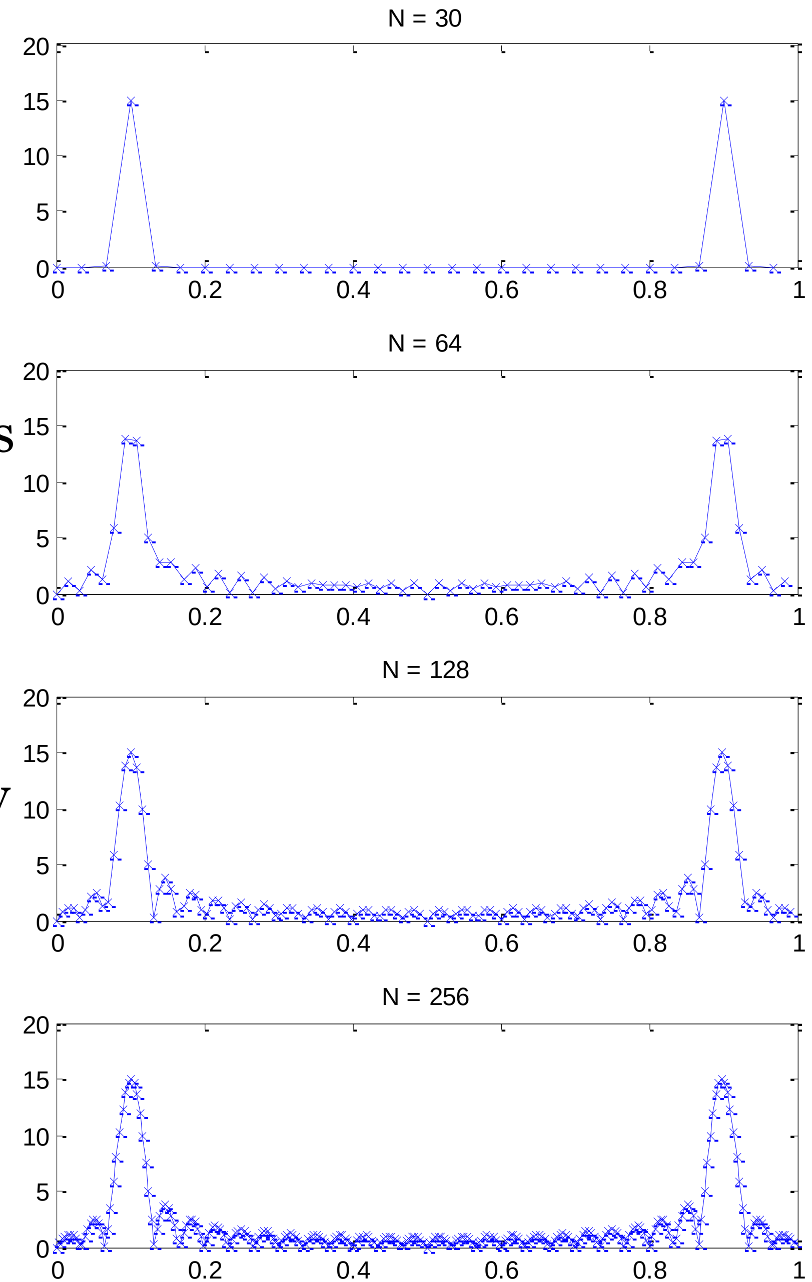
# Ex: FFT Effect of N

- Take FFT of cosine using different N values

```
n = [0:29];
x = cos(2*pi*n/10);
N1 = 64;
N2 = 128;
N3 = 256;
X1 = abs(fft(x,N1));
X2 = abs(fft(x,N2));
X3 = abs(fft(x,N3));
F1 = [0 : N1 - 1]/N1;
F2 = [0 : N2 - 1]/N2;
F3 = [0 : N3 - 1]/N3;
subplot(3,1,1)
plot(F1,X1,'-x'),title('N = 64'),axis([0 1 0 20])
subplot(3,1,2)
plot(F2,X2,'-x'),title('N = 128'),axis([0 1 0 20])
subplot(3,1,3)
plot(F3,X3,'-x'),title('N = 256'),axis([0 1 0 20])
```

- Transforms all have the same shape
- Difference is the number of samples used to approximate the shape

- Notice the sinusoid frequency is not always well represented
  - Depends on frequency resolution



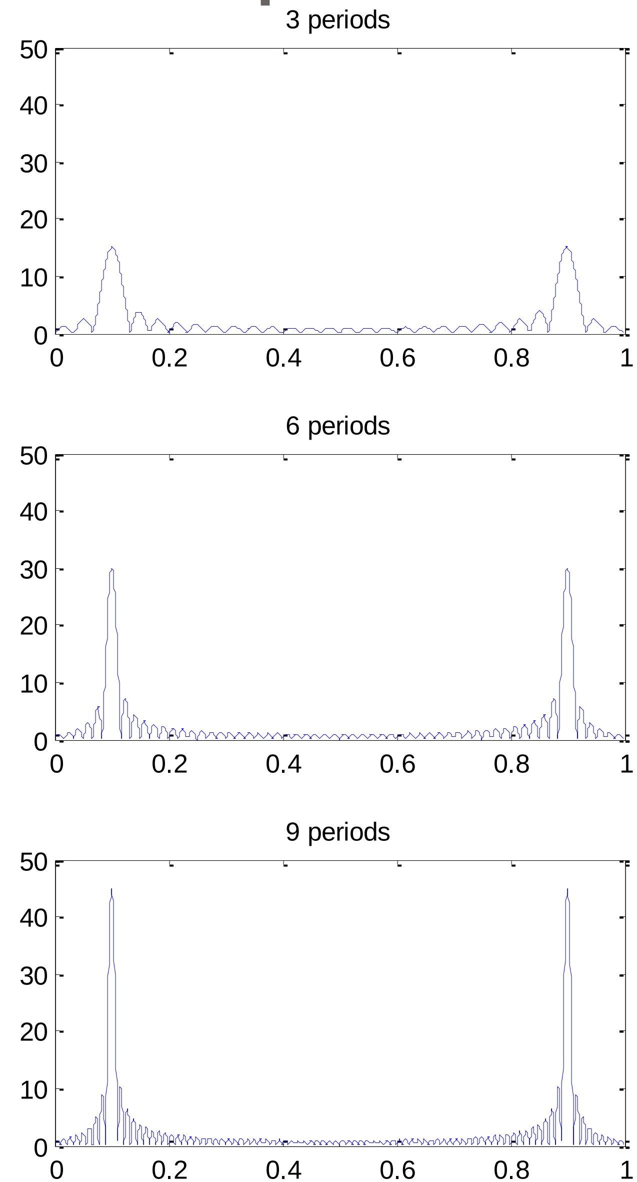
# Ex: FFT Effect of Number of Samples

- Select a large value of N and vary the number of samples of the signal

```
n = [0:29];
x1 = cos(2*pi*n/10); % 3 periods
x2 = [x1 x1]; % 6 periods
x3 = [x1 x1 x1]; % 9 periods
N = 2048;
X1 = abs(fft(x1,N));
X2 = abs(fft(x2,N));
X3 = abs(fft(x3,N));
F = [0:N-1]/N;
subplot(3,1,1)
plot(F,X1),title('3 periods'),axis([0 1 0 50])
subplot(3,1,2)
plot(F,X2),title('6 periods'),axis([0 1 0 50])
subplot(3,1,3)
plot(F,X3),title('9 periods'),axis([0 1 0 50])
```

- Transforms all have the same shape
  - Looks like sinc functions
- More samples makes the sinc look more impulse-like

- FFT with large N but fewer samples does zero-padding
  - E.g. taking length N signal and windowing with box
  - Multiplication in time is convolution in frequency



# Spectrum Analysis with FFT and Matlab

- FFT does not directly give spectrum
  - Dependent on the number of signal samples
  - Dependent on the number of points in the FFT
- FFT contains info between $[0, f_s]$
  - Spectrum must be below $f_s/2$
- Symmetric across $f = 0$ axis
  - $\left[-\frac{f_s}{2}, \frac{f_s}{2}\right]$
  - Use `fftshift.m` in Matlab

```
n = [0:149];
x1 = cos(2*pi*n/10);
N = 2048;
X = abs(fft(x1,N));
X = fftshift(X);
F = [-N/2:N/2-1]/N;
plot(F,X),
xlabel('frequency / f s')
```