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EE482: Digital Signal Processing Applications

Spring 2014
TTh 14:30-15:45 CBC C222

Lecture 05
IIR Design
14/03/04

http://www.ee.unlv.edu/~b1morris/ee482/

Outline

- Analog Filter Characteristics
- Frequency Transforms
- Design of IIR Filters
- Realizations of IIR Filters
  - Direct, Cascade, Parallel
- Implementation Considerations

IIR Design

- Reuse well studied analog filter design techniques (books and tables for design)
- Need to map between analog design and a digital design
  - Mapping between s-plane and z-plane

Analog Basics

- Laplace transform
  - $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$
- Complex s-plane
  - $s = \sigma + j\Omega$
    - Complex number with σ and Ω real
  - $j\Omega$ – imaginary axis
- Fourier transform for $\sigma = 0$
  - When region of convergence contains the $j\Omega$ axis
- Convolution relationship
  - $y(t) = x(t) * h(t) \rightarrow Y(s) = X(s)H(s)$
  - $H(s) = \frac{Y(s)}{X(s)} = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$
- Stability constraint requires poles to be in the left half s-plane

Mapping Properties

- z-transform from Laplace by change of variable
  - $z = e^{sT} = e^{\sigma T} e^{j\Omega T} = |z| e^{j\omega}$
    - $|z| = e^{\sigma T}, \quad \omega = \Omega T$
- This mapping is not unique
  - $-\pi/T < \Omega < \pi/T \;\rightarrow$ unit circle
  - 2π multiples as well

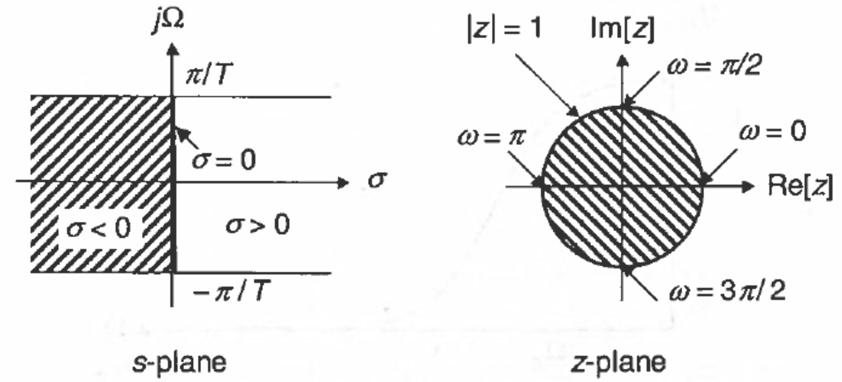

Figure 4.1 Mapping properties between the s-plane and the z-plane

  - Left half s-plane mapped inside unit circle
  - Right half s-plane mapped outside unit circle

Filter Characteristics

- Designed to meet a given/desired magnitude response
- Trade-off between :
  - Phase response
  - Roll-off rate – how steep is the transition between pass and stopband (transition width)

Butterworth Filter

- All-pole approximation to idea filter
- $|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_p)^{2L}}$
  - $|H(0)| = 1$
  - $|H(\Omega_p)| = 1/\sqrt{2}$
    - -3 dB @ Ω_p
- Has flat magnitude response in pass and stopband (no ripple)
- Slow monotonic transition band
  - Generally needs larger L

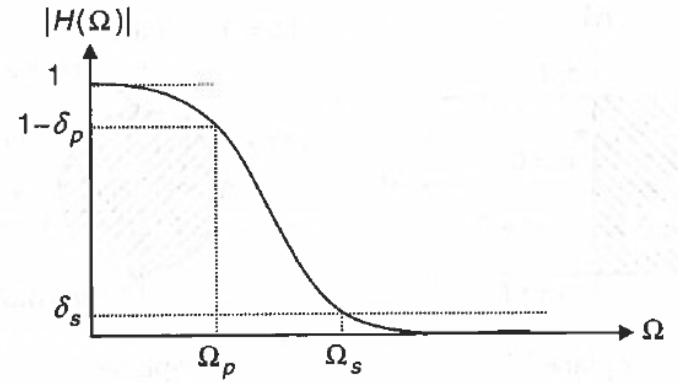

Figure 4.2 Magnitude response of Butterworth lowpass filter

Chebyshev Filter

- Steeper roll-off at cutoff frequency than Butterworth
  - Allows certain number of ripples in either passband or stopband
- Type I – equiripple in passband, monotonic in stopband
  - All-pole filter
- Type II – equiripple in stopband, monotinic in passband
  - Poles and zeros
- Generally better magnitude response than Butterworth but at cost of poorer phase response

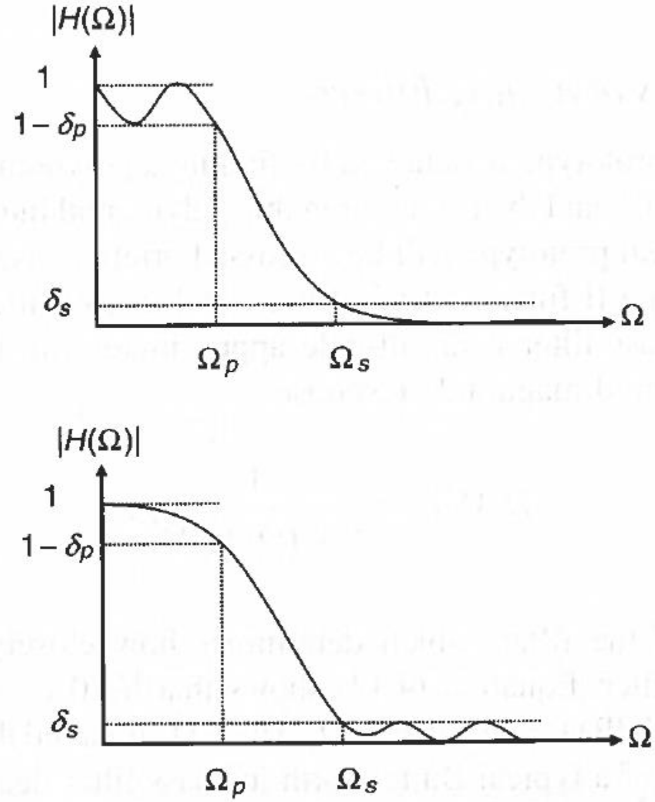

Figure 4.3 Magnitude responses of Chebyshev type I (top) and type II lowpass filters

Elliptic Filter

- Sharpest passband to stopband transition
- Equiripple in both pass and stopbands
- Phase response is highly unlinear in passband
  - Should only be used in situations where phase is not important to design

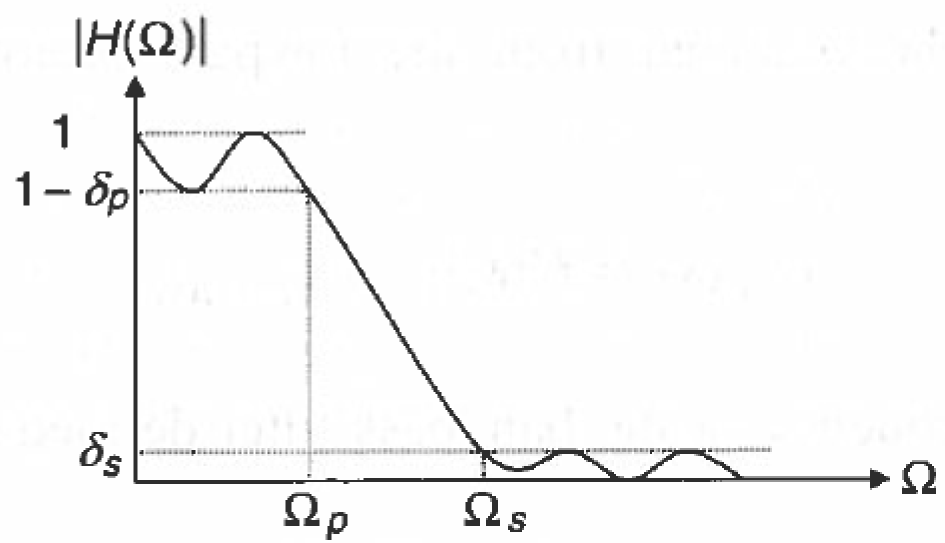

Figure 4.4 Magnitude response of elliptic lowpass filter

Frequency Transforms

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- Design lowpass filter and transform from LP to another type (HP, BP, BS)

- Define mapping
- $H(z) = H_{lp}(Z)|_{Z^{-1}=G(z^{-1})}$
  - Replace Z^{-1} in LP filter with $G(z^{-1})$

- θ – frequency in LP
- ω – frequency in new transformed filter

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

IIR Filter Design

- IIR transfer function

$$H(z) = \frac{\sum_{l=0}^{L-1} b_l z^{-l}}{1 + \sum_{l=0}^{M} a_l z^{-l}}$$

- Need to find coefficients a_l, b_l
  - Impulse invariance – sample impulse response
    - Have to deal with aliasing
  - Bilinear transform
    - Match magnitude response
    - "Warp" frequencies to prevent aliasing

Bilinear Transform Design

- Convert digital filter into an “equivalent” analog filter
  - Use bilinear “warping”
- Design analog filter using IIR design techniques
- Map analog filter into digital
  - Use bilinear transform

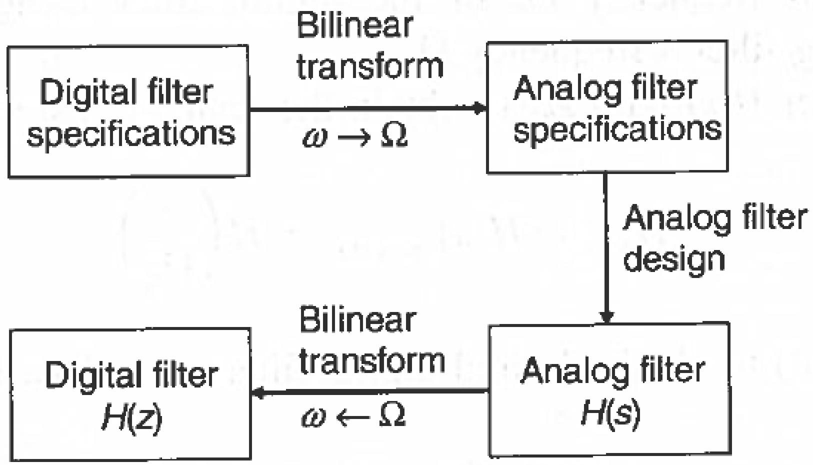


Figure 4.5 Digital IIR filter design using the bilinear transform

Bilinear Transformation

- Mapping from s-plane to z-plane
- $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$
- Frequency mapping
  - $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$
  - $\omega = 2\arctan\left(\frac{\Omega T}{2}\right)$
- Entire $j\omega$-axis is squished into $[-\pi/T, \pi/T]$ to prevent aliasing
  - Unique mapping
  - Highly non-linear which requires "pre-warp" in design

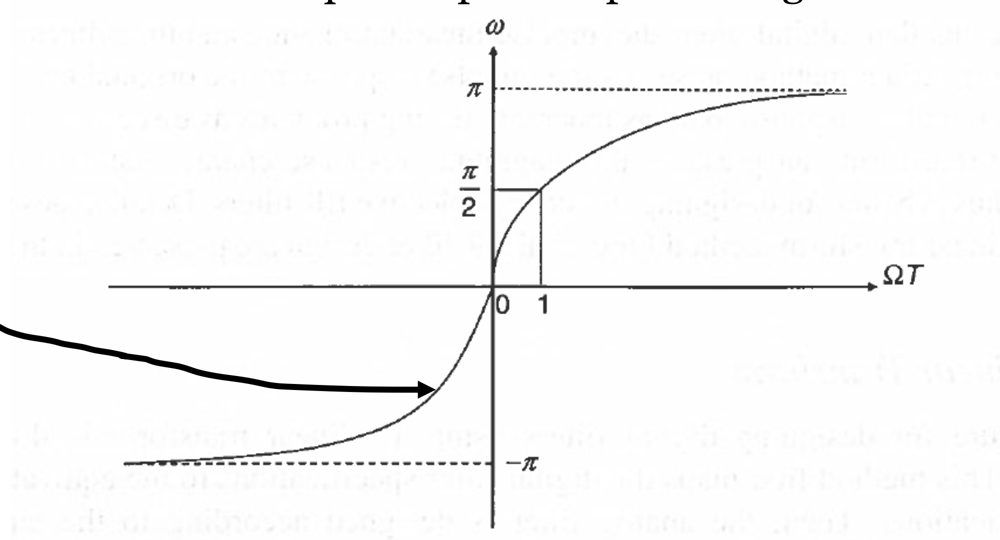
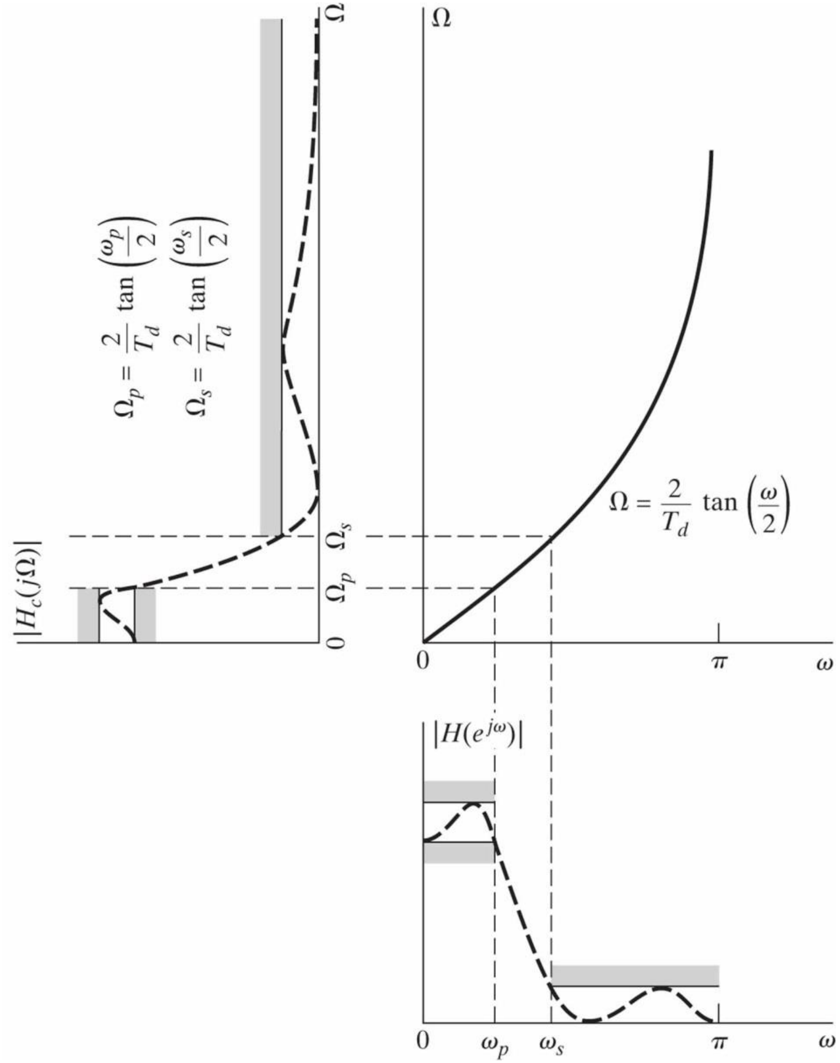


Figure 4.6 Frequency warping of bilinear transform defined by (4.27)

Bilinear Design Steps

1. Convert digital filter into an "equivalent" analog filter
   - Pre-warp using
     - $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$
2. Design analog filter using IIR design techniques
   - Butterworth, Chebyshev, Elliptical
3. Map analog filter into digital
   - $H(z) = H(s)|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$



Bilinear Design Example

- Example 4.2
- Design filter using bilinear transform
  - $H(s) = 1/(s+1)$
  - Bandwith 10000 Hz
  - $f_s = 8000$ Hz
- Parameters
  - $\omega_c = 2\pi(1000/8000) = 0.25\pi$

1. Pre-warp
   - $\Omega_c = \frac{2}{T}\tan(0.125\pi) = \frac{0.8284}{T}$
2. Scale frequency (normalize scale)
   - $\hat{H}(s) = H\left(\frac{s}{\Omega_c}\right) = \frac{0.8284}{sT+0.8284}$
3. Bilinear transform
   - $H(z) = \frac{0.2929(1+z^{-1})}{1-0.4141z^{-1}}$

IIR Filter Realizations

- Different forms or structures can implement an IIR filter
  - All are equivalent mathematically (infinite precision)
  - Different practical behavior when considering numerical effects
- Want structures to minimize error

Direct Form I

- Straight-forward implementation of diff. eq.
  - b_l - feed forward coefficients
    - From $x(n)$ terms
  - a_l - feedback coefficients
    - From $y(n)$ terms
- Requires $(L + M)$ coefficients and delays

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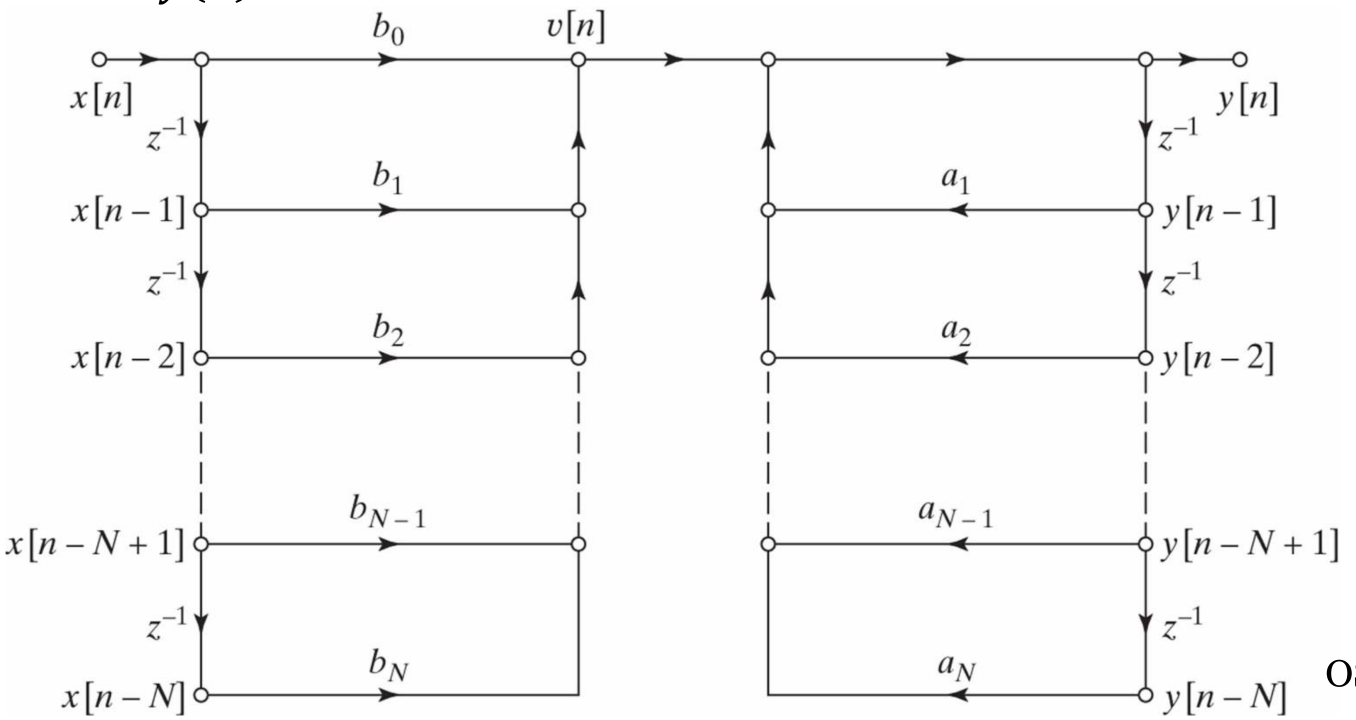

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Direct Form II

- Notice that we can decompose the transfer function
  - $H(z) = H_1(z)H_2(z)$
    - Section to implement zeros section to implement poles
- Can switch order of operations
  - $H(z) = H_2(z)H_1(p)$
  - This allows sharing of delays and saving in memory

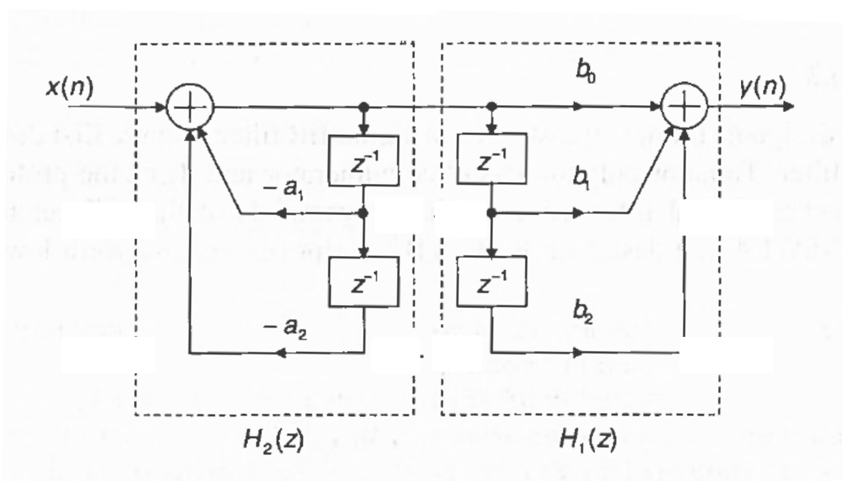
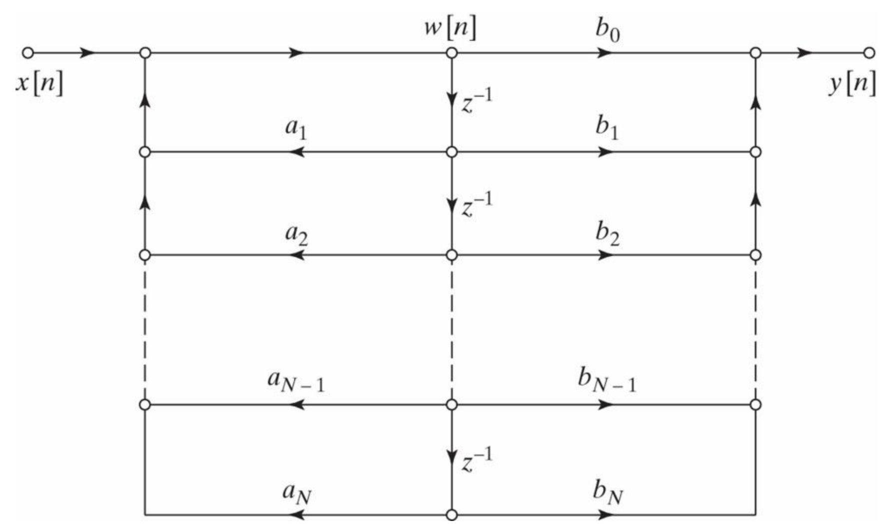


Figure 4.7 Direct-form I realization of second-order IIR filter



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Cascade (Factored) Form

- Factor transfer function and decompose into smaller sub-systems
  - $H(z) = H_1(z)H_2(z) \dots H_K(z)$

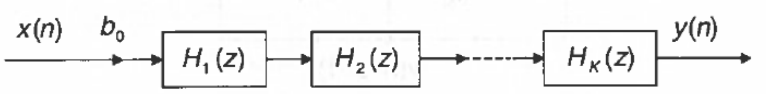
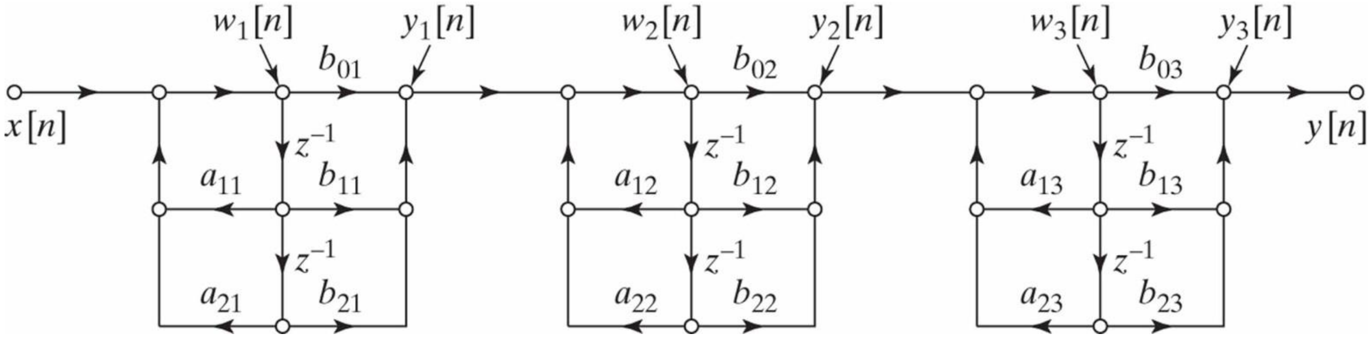


Figure 4.10 Cascade realization of digital filter

- Make each subsystem second order
  - Complex conjugate roots have real coefficients
  - Limit the order of subsystem (numerical effects)
    - Effects limited to single subsystem stage
    - Change in a single coefficient affects all poles in DF



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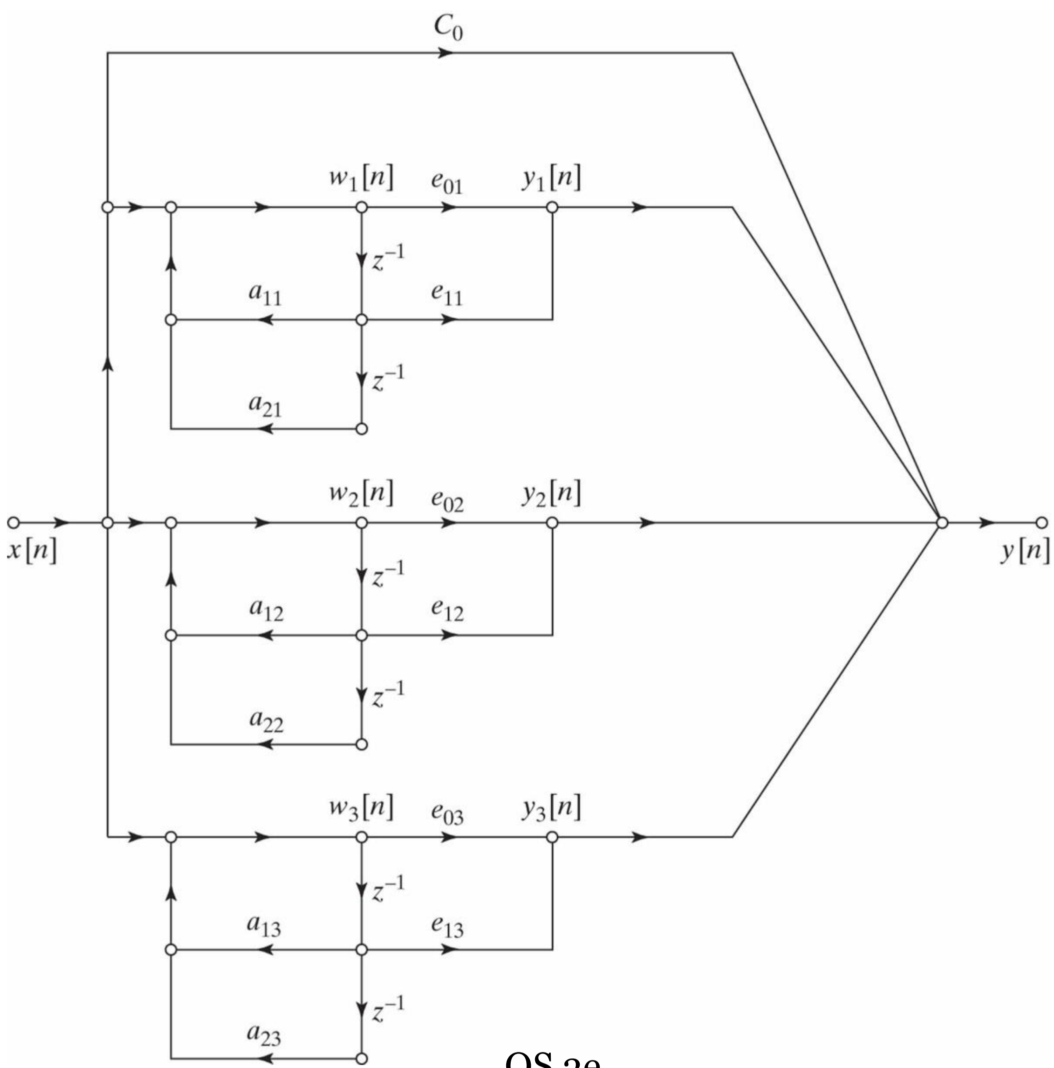
- Preferred over DF because of numerical stability

Parallel (Partial Fraction) Form

- Decompose transfer function using a partial fraction expansion
  - $H(z) = H_1(z) + H_2(z) + \ \ldots + H_K(z)$
    - $H_k(z) = \frac{b_{0k}+b_{1k}z^{-1}}{1+a_{1k}z^{-1}+a_{2k}z^{-2}}$
- Be sure to remember that PFE requires numerator order less than denominator
  - Use polynomial long division



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