EE482: DSP APPLICATIONS

CH2 DSP FUNDAMENTALS

http://www.ee.unlv.edu/~b1morris/ee482
DIGITAL SIGNALS AND SYSTEMS

CHAPTER 2.1
ELEMENTARY DIGITAL SIGNALS

- Digital signal
  - $x(n) \quad n \in \mathbb{Z}$
  - Deterministic – expressed mathematically (e.g. sinusoid)
  - Random – cannot be described exactly by equations (e.g. noise, speech)
- Unit impulse (Kronecker delta)
  - $\delta(n) = \begin{cases} 
  1, & n = 0 \\
  0, & n \neq 0 
\end{cases}$
  - Basic building block of all digital signals
- Unit step
  - $u(n) = \begin{cases} 
  1, & n \geq 0 \\
  0, & n < 0 
\end{cases} = \Sigma_{k=-\infty}^{n} \delta(k)$
SINUSOIDAL SIGNALS

- Continuous
  - \( x(t) = A \sin(\Omega t + \phi) = A \sin(2\pi f t + \phi) \)

- Sampled
  - \( x(n) = A \sin(\Omega n T + \phi) = A \sin(2\pi f n T + \phi) \)
  - \( \Omega = 2\pi f \)
  - \( x(n) = A \sin(\omega n + \phi) = A \sin(F\pi n + \phi) \)
  - \( \omega = \Omega T \)
### Table 2.1

<table>
<thead>
<tr>
<th>Variable</th>
<th>Units</th>
<th>Relationships</th>
<th>Ranges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega$</td>
<td>rads/sec</td>
<td>$\Omega = 2\pi f$</td>
<td>$-\infty &lt; \Omega &lt; \infty$</td>
</tr>
<tr>
<td>$f$</td>
<td>cycles/sec (Hz)</td>
<td>$f = \frac{\Omega}{2\pi} = \frac{\omega f_s}{2\pi}$</td>
<td>$-\infty &lt; f &lt; \infty$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>rads/sample</td>
<td>$\omega = \Omega T = \frac{2\pi f}{f_s}$</td>
<td>$-\pi \leq \omega \leq \pi$</td>
</tr>
<tr>
<td>$F$</td>
<td>cycles/sample</td>
<td>$F = \frac{f}{f_s/2} = \frac{\omega}{2}$</td>
<td>$-1 \leq F \leq 1$</td>
</tr>
</tbody>
</table>

- Normalized frequency measures
- Note: max frequency for $\pi$ or definition over a $2\pi$ interval
  - Consider $e^{j(\omega+2\pi k)}$
EXAMPLE 2.1

- $A=2$; $f=1000$; $fs=8000$;
- $n=0:31$;
- $w = 2\pi f/fs$;
- $x = A\sin(w*n)$;
- $h=figure$;
- %plot sampled sine
- subplot(2,1,1)
- plot(n,x,'-o','linewidth',2);
- xlabel('time index $[n]$'); ylabel('value'); legend('sampled sine')
- %plot analog sine
- subplot(2,1,2)
- $t=0:1e-5:4e-3$;
- plot(n*(1/fs),x,'-o','linewidth',2);
- hold all;
- plot(t, $A\sin(2\pi f*t)$, 'linewidth',2);
- xlabel('time $[t=n(1/f_s)]$'); ylabel('value'); legend('sampled sine', 'analog sine')
Processing accomplished with 3 basic operations

- **Addition**
  \[ y(n) = x_1(n) + x_2(n) \]

- **Multiplication**
  \[ y(n) = ax(n) \]

- **Time shift (delay)**
  \[ y(n) = x(n - L) \]

Multiple delays can be implemented with a shift register (first-in, first-out buffer)(tapped delay line)

**Multiplication in z domain**
SYSTEM CONCEPTS

CHAPTER 2.2
Generic system

\[ x(n) \xrightarrow{T} y(n) \]

Linearity

- Additive and homogeneity (scaling) properties
  \[ T\{ax_1(n) + bx_2(n)\} = ay_1(n) + by_2(n) \]

Time invariance

- Shift in input causes corresponding shift in output
  \[ y(n - n_0) = T\{x(n - n_0)\} \]

- To test, check if \( y_1(n) = y_2(n) \)
  - \( y_1(n) = y(n - n_0) \)
    - Replace \( n \) by \( n_0 \)
  - \( y_2(n) = T\{x(n - n_0)\} = T\{g(n)\} \)
    - Response of system to shifted input
LTI SYSTEMS

- Impulse response

\[ x(n) = \delta(n) \xrightarrow{LTI} y(n) = h(n) \]

- Output of LTI system \( y(n) = h(n) \) to input \( x(n) = \delta(n) \)

- Convolution

- Input-output relationship of LTI system

\[ y(n) = x(n) \ast h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \]
GENERAL DIFFERENCE EQUATION SYSTEMS

\[ \sum_{k=0}^{M} a_k y(n-k) = \sum_{k=0}^{L-1} b_k x(n-k) \]

- Infinite impulse response (IIR)
  - \( h(n) \) non-zero as \( n \to \infty \)
- Finite impulse response (FIR)
  - \( h(n) \) defined over finite set of \( n \)
  - Special case of above with \( a_k = 0 \)
  - This system only has zeroes and poles at \( z = 0 \)

\[ y(n) = \sum_{k=0}^{L-1} b_k x(n-k) - \sum_{k=1}^{M} a_k y(n-k) \]

- Causality
  - Output only depends on previous input
    - \( h(n) = 0, \quad n < 0 \)
- Stability (BIBO)
  - \( \sum \vert h(n) \vert < \infty \)
  - Absolutely summable
Z-TRANSFORM

- Very useful computational tool for studying digital systems

- Definition \( X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad z = re^{j\theta} \)

- Has associated region of convergence (ROC)
  - Values of \( z \) where summation converges

- Useful summation formulas
  \[
  \sum_{n=0}^{N} \alpha^n = \frac{1 - \alpha^{N+1}}{1 - \alpha} \quad \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1 - \alpha} \quad |\alpha| < 1
  \]
Z-TRANSFORM PROPERTIES

- **Linearity**
  \[ \mathcal{Z}\{ax_1(n) + bx_2(n)\} = aX_1(z) + bX_2(z) \]

- **Time shift**
  \[ \mathcal{Z}\{x(n - k)\} = z^{-k}X(z) \]

- **Convolution**
  \[ x(n) = x_1(n) * x_2(n) \rightarrow X(z) = X_1(z)X_2(z) \]
  \[ \text{ROC} = R_{x_1} \cap R_{x_2} \]
Note: convolution in time is multiplication in Z-domain

\[ Y(z) = X(z)H(z) \]

\[ H(z) = \frac{Y(z)}{X(z)} \]

General polynomial form from difference equation

- Take Z-transform of both sides of general diff eq

\[ H(z) = \frac{\sum_{k=0}^{L-1} b_k z^{-k}}{1 + \sum_{k=1}^{M} a_k z^{-k}} \]
POLES AND ZEROS

- **Zeros**
  - Roots of the numerator polynomial
  - Locations in z-plane that make output zero

- **Poles**
  - Roots of the denominator polynomial
  - Locations in z-plane that make output infinity (unstable)
    - System is considered unstable if the ROC doesn’t contain the unit circle (no DTFT exists)
    - Causal system → poles should be inside unit circle

\[
H(z) = \frac{\sum_{k=0}^{L-1} b_k z^{-k}}{1 + \sum_{k=1}^{M} a_k z^{-k}} \quad \rightarrow \quad H(z) = b_0 \frac{\prod_{k=1}^{L-1} (z - z_k)}{\prod_{k=1}^{M} (z - p_k)} = b_0 \frac{(z - z_1)(z - z_2)\ldots}{(z - p_1)(z - p_2)\ldots}
\]
EXAMPLE 2.10

- $H(z) = \frac{1}{L} \left[ \frac{1-z^{-L}}{1-z^{-1}} \right]$
  - Notice this is a polynomial in $z^{-1}$
- Convert to polynomial in $z$ to get all poles and zeros
  
  $H(z) = \frac{1}{L} \left[ \frac{z^L-1}{z^L-z^{-1}} \right] = \frac{1}{L} \left[ \frac{z^L-1}{z^{L-1}(z-1)} \right]$
  - Poles
    - $(z - 1) = 0 \rightarrow z = 1$
    - $z^{L-1} = 0 \rightarrow L-1$ poles at $z = 0$
  - Zeros
    - $z^L - 1 = 0 \rightarrow z_l = e^{j(\frac{2\pi}{L})l}$
      - $L$ zeros even spaced around unit circle

Matlab

```
fvtool([1 0 0 0 0 0 0 -1], [1 -1]);
```

Figure 2.12  Pole–zero diagram of the moving-averaging filter, $L = 8$
Discrete-time Fourier transform (DTFT)

\[ H(\omega) = H(z) \bigg|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \]

- Evaluate transfer function along the unit circle \(|z| = |e^{j\omega}|\)

\[ H(\omega) = |H(\omega)|e^{j\angle H(\omega)} \]

\[ |H(\omega)| = \sqrt{H(\omega)H^*(\omega)} \quad \angle H(\omega) = \arctan \left( \frac{\text{Im} H(\omega)}{\text{Re} H(\omega)} \right) \]

- Frequency response is periodic in \(2\pi\) interval and symmetric
  - Only \([0, \pi]\) interval is required for evaluation
Poles
- $|H(\omega)|$ gets larger closer to $\theta$

Zeros
- $|H(\omega)|$ gets smaller closer to $\theta$

- What does a highpass filter look like?

- What does a lowpass filter look like?
Notice the DTFT is a continuous function of $\omega$
- Requires an infinite number of samples to compute (infinite sum)

DFT is a sampled version of the DTFT
- Samples are taken at $N$ equally spaced frequencies along unit circle
  - $\omega_k = \frac{2\pi k}{N}, \; k = 0, 1, ..., N - 1$

\[ X(k) = X(\omega)|_{\omega = \frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi k}{N}n} \]

- $n$ – time index
- $k$ – frequency index
DFT IMPLEMENTATION

$$X(k) = X(\omega)\big|_{\omega = \frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi k}{N}n}$$

- DFT can be computed very efficiently with the fast Fourier transform (FFT)
- Frequency resolution of DFT
  - $\Delta\omega = \frac{2\pi}{N}$, $\Delta f = \frac{f_s}{N}$
- Analog frequency mapping
  - $f_k = k\Delta f = \frac{kf_s}{N}$, $k = 0, 1, ..., N - 1$
  - Nyquist frequency $\frac{f_s}{2}$ corresponds to $k = \frac{N}{2}$
EXAMPLE 2.16

- \( N = 100; \ A = 1; \ f=1000; \ fs = 10000; \)
- \( n=0:N-1; \)
- \( w = 2\pi f/fs; \)
- \( x = \sin(w*n); \)
- \( X = \text{fft}(x); \)
- \( K = \text{length}(X); \)
- \( h=\text{figure}; \)
- \( \text{subplot}(2,1,1) \)
  - \( \text{plot}(0:K-1, 20*\log10(\text{abs}(X)), \ 'linewidth', \ 2); \)
  - \( \text{xlabel}('freq index [k]'); \text{ylabel}('magnitude [dB]'); \)
  - \( \text{subplot}(2,1,2) \)
  - %convert index to freq
  - \( f = (0:K-1) * fs/N; \)
  - \( \text{plot}(f, 20*\log10(\text{abs}(X)), \ 'linewidth', \ 2); \)
  - \( \text{xlabel}('freq [Hz]'); \text{ylabel}('magnitude [dB]'); \)
EXAMPLE 2.16 – SHIFTED FREQUENCIES

- \( \text{ind} = -K/2:K/2-1; \)
- \( \text{Xs} = \text{fftshift}(X) \)
- \( h=\text{figure}; \)
- \( \text{subplot}(2,1,1) \)
- \( \text{plot}(\text{ind}, 20\cdot\log10(\text{abs}(\text{Xs})), \text{'}linewidth\text{'}, 2); \)
- \( \text{xlabel}(\text{'}freq \text{ index [k]}\text{');} \)
- \( \text{ylabel}(\text{'}magnitude [dB]\text{');} \)
- \( \text{subplot}(2,1,2) \)
- \( \) %convert index to freq
- \( f = \text{ind} \times \text{fs}/N; \)
- \( \text{plot}(f, 20\cdot\log10(\text{abs}(\text{Xs})), \text{'}linewidth\text{'}, 2); \)
- \( \text{xlabel}(\text{'}freq [Hz]\text{');} \)
- \( \text{ylabel}(\text{'}magnitude [dB]\text{');} \)
- \( \)
INTRO TO RANDOM VARIABLES

CHAPTER 2.3
RANDOM VARIABLES

- Function that maps from a sample space to a real value
  - \( x: S \rightarrow \mathbb{R} \)
  - \( x \) – random variable (does not have a value)
  - \( S \) – sample space

- Cumulative distribution function (CDF)
  - \( F(X) = P(x \leq X) \)
  - E.g. probability \( \{x \leq X\} \)

- Probability density function (PDF)
  - \( f(X) = \frac{dF(X)}{dx} \)
  - \( \int_{-\infty}^{\infty} f(X)dX = 1 \)
  - \( P(X_1 < x \leq X_2) = F(X_2) - F(X_1) \)
  - \( P(X_1 < x \leq X_2) = \int_{X_1}^{X_2} f(X)dX \)

- Probability mass function (PMF)
  - For discrete \( x \), takes values \( X_i, \ i = 1, 2, 3, \ldots \)
  - \( p_i = P(x = X_i) \)
UNIFORM RANDOM VARIABLE

- Variable takes on value in a range with equal probability

- \[ f(X) = \begin{cases} \frac{1}{X_2 - X_1} & X_1 \leq x \leq X_2 \\ 0 & \text{else} \end{cases} \]

- Be sure you can calculate mean and variance

- Be aware that the book is a little sloppy in notation
  - RV \( x \) vs \( X \)

Figure 2.17  The uniform density function
STATISTICS OF RANDOM VARIABLES

- **Expected value (mean)**
  - \( m_x = E[x] \) expectation operator
    - Continuous
      \[ m_x = \int_{-\infty}^{\infty} x f(x) dX \]
    - Discrete
      \[ m_x = \sum_i x_i p_i \]

- Can be computed with \texttt{mean.m} and \texttt{var.m}
  - Read help for info on finite sample versions

- **Variance (spread around mean)**
  - \( \sigma_x^2 = E[(x - m_x)^2] = E[x^2] - m_x^2 \)
    - Continuous
      \[ \sigma_x^2 = \int_{-\infty}^{\infty} (x - m_x)^2 f(x) dX \]
    - Discrete
      \[ \sigma_x^2 = \sum_i p_i (X_i - m_x)^2 \]
  - For \( m_x = 0 \),
    \[ \sigma_x^2 = E[x^2] = P_x \]
    - Second moment, power
FIXED-POINT REPRESENTATION AND QUANTIZATION EFFECTS

CHAPTER 2.4
Fractional numbers are represented in 2’s complement with $B = M + 1$ bits

$x = b_0.b_1b_2...b_{M-1}b_M$

- $b_0 = \begin{cases} 0 & x \geq 0 \text{ positive} \\ 1 & x < 0 \text{ negative} \end{cases}$

- Value = $-b_0 + \sum_{m=1}^{M} b_m 2^{-m}$
  - $-1 \leq x \leq (1 - 2^{-M})$
  - Unbalanced range with more negative than positive numbers
GENERAL FRACTIONAL FORMAT $Q_{n.m}$

- $x = b_0 b_1 b_2 ... b_n \cdot b_1 b_2 ... b_M$
  
  - $b_0$ is not counted as part of integer just as a sign-bit

- $Q$ format
  
  - $Q_{n.m} = Q^n \#integer.\#fraction$
  
  - Larger $n$ increases dynamic range but at cost of reduced precision (smallest fractional resolution)

- Example 2.25
  
  - $x = 0100 1000 0001 1000b$
    
    $= 0x4818$

  - $Q_{0.15}$
    
    - $x = 2^{-1} + 2^{-4} + 2^{-11} + 2^{-12} = 0.56323$

  - $Q_{2.13}$
    
    - $x = 2^1 + 2^{-2} + 2^{-9} + 2^{-10} = 2.25293$

  - $Q_{5.10}$
    
    - $x = 2^4 + 2^1 + 2^{-6} + 2^{-7} = 18.02344$
FINITE WORD LENGTH EFFECTS

1. Quantization errors
   - Signal quantization
   - Coefficient quantization

2. Arithmetic errors
   - Roundoff (truncation)
   - Overflow
ADC conversion of sampled signals to fixed levels

- Using Q15 and $B$ bits
  - Dynamic range $-1 \leq x < 1$
  - Quantization step
    - $\Delta = \frac{2}{2^B} = 2^{-B+1} = 2^{-M}$

- Quantization error
  - $e(n) = x(n) - x_B(n)$
  - $x_B(n) = Q[x(n)]$
  - $|e(n)| \leq \frac{\Delta}{2} = 2^{-B}$ (rounding)
    - Error dependent on word length $B$
    - More bits for better resolution, less error (noise)

- Signal to quantization noise (SQNR)
  - $SQNR = \frac{\sigma_x^2}{\sigma_e^2} = 3.2^{2B} \sigma_x^2$
  - $SQNR = 4.77 + 6.02B + 10 \log_{10} \sigma_x^2 \, dB$

**Figure 2.21** Quantization process related to a 3-bit ADC
COEFFICIENT QUANTIZATION

- Same error issues as for signals
- Results in movement of the locations of poles/zeros
  - Changes system function polynomials
  - Can lead to instability if poles go outside the unit circle
    - Generally, more a problem with IIR filters
- Can limit coefficient quantization effects by using lower-order filters
  - Use of cascade and parallel filter structures
ROUND OFF NOISE

- A product must be represented in $B$ bits by rounding (truncation)
  - $y(n) = ax(n)$

- Error model
  - $y(n) = Q[ax(n)] = ax(n) + e(n)$
    - $e(n)$ is uniformly distributed zero mean noise (rounding)
OVERFLOW AND SOLUTIONS

CHAPTER 2.5
OVERFLOW

- \( y(n) = x_1(n) + x_2(n) \)
  - \(-1 \leq x_i(n) < 1\)
  - \(-1 \leq y(n) < 1\)
- Overflow occurs when the sum cannot fit in the word container
- Signals need to be scaled to prevent overflow

- Notice: this reduces the SQNR
  - \( SQNR = 10 \log_{10} \left( \frac{\beta^2 \sigma_x^2}{\sigma_e^2} \right) \)
  - \( SQNR = 4.77 + 6.02B + 10 \log_{10} \sigma_x^2 + 20 \log_{10} \beta \) dB

Negative since \( \beta < 1 \)