EE482: Digital Signal Processing Applications

Spring 2014
TTh 14:30-15:45 CBC C222

Lecture 05
IIR Design
14/03/04

http://www.ee.unlv.edu/~b1morris/ee482/
Outline

• Analog Filter Characteristics
• Frequency Transforms
• Design of IIR Filters
• Realizations of IIR Filters
  ▫ Direct, Cascade, Parallel
• Implementation Considerations
IIR Design

- Reuse well studied analog filter design techniques (books and tables for design)
- Need to map between analog design and a digital design
  - Mapping between s-plane and z-plane
Analog Basics

- Laplace transform
  \[ X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt \]
- Complex s-plane
  \[ s = \sigma + j\Omega \]
  - Complex number with \( \sigma \) and \( \Omega \) real
  - \( j\Omega \) – imaginary axis
- Fourier transform for \( \sigma = 0 \)
  - When region of convergence contains the \( j\Omega \) axis
- Convolution relationship
  \[ y(t) = x(t) \ast h(t) \rightarrow Y(s) = X(s)H(s) \]
  \[ H(s) = \frac{Y(s)}{X(s)} = \int_{-\infty}^{\infty} h(t)e^{-st} \, dt \]
- Stability constraint requires poles to be in the left half s-plane
Mapping Properties

- z-transform from Laplace by change of variable
  \[ z = e^{sT} = e^{\sigma T} e^{j\Omega T} = |z|e^{j\omega} \]
  \[ |z| = e^{\sigma T}, \quad \omega = \Omega T \]

- This mapping is not unique
  - \(-\pi/T < \Omega < \pi/T \rightarrow\) unit circle
  - \(2\pi\) multiples as well

![Diagram](image)

**Figure 4.1**  Mapping properties between the s-plane and the z-plane

- Left half s-plane mapped inside unit circle
- Right half s-plane mapped outside unit circle
Filter Characteristics

- Designed to meet a given/desired magnitude response
- Trade-off between:
  - Phase response
  - Roll-off rate – how steep is the transition between pass and stopband (transition width)
Butterworth Filter

- All-pole approximation to idea filter
- $|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_p)^{2L}}$
  - $|H(0)| = 1$
  - $|H(\Omega_p)| = 1/\sqrt{2}$
    - -3 dB @ $\Omega_p$
- Has flat magnitude response in pass and stopband (no ripple)
- Slow monotonic transition band
  - Generally needs larger $L$

![Magnitude response of Butterworth lowpass filter](image)
Chebyshev Filter

- Steeper roll-off at cutoff frequency than Butterworth
  - Allows certain number of ripples in either passband or stopband
- Type I – equiripple in passband, monotonic in stopband
  - All-pole filter
- Type II – equiripple in stopband, monotonic in passband
  - Poles and zeros
- Generally better magnitude response than Butterworth but at cost of poorer phase response

Figure 4.3 Magnitude responses of Chebyshev type I (top) and type II lowpass filters
Elliptic Filter

- Sharpest passband to stopband transition
- Equiripple in both pass and stopbands
- Phase response is highly unlinear in passband
  - Should only be used in situations where phase is not important to design

Figure 4.4  Magnitude response of elliptic lowpass filter
Frequency Transforms

- Design lowpass filter and transform from LP to another type (HP, BP, BS)
  - Define mapping
  - $H(z) = H_{lp}(Z)|_{z^{-1}=G(z^{-1})}$
  - Replace $z^{-1}$ in LP filter with $G(z^{-1})$
  - $\theta$ – frequency in LP
  - $\omega$ – frequency in new transformed filter

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Transformations</th>
<th>Associated Design Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lowpass</td>
<td>$z^{-1} = \frac{z^{-1} - \alpha}{1 - az^{-1}}$</td>
<td>$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ $\omega_p = \text{desired cutoff frequency}$</td>
</tr>
<tr>
<td>Highpass</td>
<td>$z^{-1} = \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$</td>
<td>$\alpha = -\frac{\cos\left(\frac{\omega_p + \omega_p}{2}\right)}{\cos\left(\frac{\omega_p - \omega_p}{2}\right)}$ $\omega_p = \text{desired cutoff frequency}$</td>
</tr>
<tr>
<td>Bandpass</td>
<td>$z^{-1} = \frac{z^{-2} - \frac{2ak}{k+1}z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2} - \frac{2ak}{k+1}z^{-1} + 1}$</td>
<td>$\alpha = \frac{\cos\left(\frac{\omega_p^2 + \omega_p^1}{2}\right)}{\cos\left(\frac{\omega_p^2 - \omega_p^1}{2}\right)}$ $\omega_p^1 = \text{desired lower cutoff frequency}$ $\omega_p^2 = \text{desired upper cutoff frequency}$</td>
</tr>
<tr>
<td>Bandstop</td>
<td>$z^{-1} = \frac{z^{-2} - \frac{2a}{1+k}z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k}z^{-2} - \frac{2a}{1+k}z^{-1} + 1}$</td>
<td>$\alpha = \frac{\cos\left(\frac{\omega_p^2 + \omega_p^1}{2}\right)}{\cos\left(\frac{\omega_p^2 - \omega_p^1}{2}\right)}$ $\omega_p^1 = \text{desired lower cutoff frequency}$ $\omega_p^2 = \text{desired upper cutoff frequency}$</td>
</tr>
</tbody>
</table>
IIR Filter Design

- IIR transfer function

\[ H(z) = \frac{\sum_{l=0}^{L-1} b_l z^{-l}}{1 + \sum_{l=0}^{M} a_l z^{-l}} \]

- Need to find coefficients \( a_l, b_l \)
  - Impulse invariance – sample impulse response
    - Have to deal with aliasing
  - Bilinear transform
    - Match magnitude response
    - “Warp” frequencies to prevent aliasing
Bilinear Transform Design

- Convert digital filter into an “equivalent” analog filter
  - Use bilinear “warping”
- Design analog filter using IIR design techniques
- Map analog filter into digital
  - Use bilinear transform

![Diagram](image)

**Figure 4.5** Digital IIR filter design using the bilinear transform
Bilinear Transformation

- Mapping from s-plane to z-plane
  
  \[ s = \frac{2}{T} \left( \frac{z-1}{z+1} \right) = \frac{2}{T} \left( \frac{1-z^{-1}}{1+z^{-1}} \right) \]

- Frequency mapping
  
  \[ \Omega = \frac{2}{T} \tan \left( \frac{\omega}{2} \right) \]
  
  \[ \omega = 2 \arctan \left( \frac{\Omega T}{2} \right) \]

- Entire j\omega-axis is squished into \([-\pi/T, \pi/T]\) to prevent aliasing
  
  - Unique mapping
  
  - Highly non-linear which requires “pre-warp” in design

Figure 4.6 Frequency warping of bilinear transform defined by (4.27)
Bilinear Design Steps

1. Convert digital filter into an “equivalent” analog filter
   - Pre-warp using
     \[ \Omega = \frac{2}{T} \tan \left( \frac{\omega}{2} \right) \]

2. Design analog filter using IIR design techniques
   - Butterworth, Chebyshev, Elliptical

3. Map analog filter into digital
   - \[ H(z) = \left| H(s) \right| \bigg|_{s = \frac{2}{T} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right)} \]
Bilinear Design Example

- Example 4.2
- Design filter using bilinear transform
  - $H(s) = 1/(s + 1)$
  - Bandwidth 10000 Hz
  - $f_s = 8000$ Hz
- Parameters
  - $\omega_c = 2\pi (1000/8000) = 0.25\pi$

1. Pre-warp
   - $\Omega_c = \frac{2}{T} \tan(0.125\pi) = \frac{0.8284}{T}$

2. Scale frequency (normalize scale)
   - $\hat{H}(s) = H\left(\frac{s}{\Omega_c}\right) = \frac{0.8284}{sT+0.8284}$

3. Bilinear transform
   - $H(z) = \frac{0.2929(1+z^{-1})}{1-0.4141z^{-1}}$
IIR Filter Realizations

• Different forms or structures can implement an IIR filter
  ▫ All are equivalent mathematically (infinite precision)
  ▫ Different practical behavior when considering numerical effects

• Want structures to minimize error
Direct Form I

- Straight-forward implementation of diff. eq.
  - $b_l$ - feed forward coefficients
    - From $x(n)$ terms
  - $a_l$ - feedback coefficients
    - From $y(n)$ terms
- Requires $(L + M)$ coefficients and delays

OS 3e
Direct Form II

- Notice that we can decompose the transfer function
  - \( H(z) = H_1(z)H_2(z) \)
  - Section to implement zeros
  - Section to implement poles

- Can switch order of operations
  - \( H(z) = H_2(z)H_1(p) \)
  - This allows sharing of delays and saving in memory

Figure 4.7  Direct-form I realization of second-order IIR filter
Cascade (Factored) Form

• Factor transfer function and decompose into smaller sub-systems
  - \( H(z) = H_1(z)H_2(z) \ldots H_K(z) \)

• Make each subsystem second order
  - Complex conjugate roots have real coefficients
  - Limit the order of subsystem (numerical effects)
    - Effects limited to single subsystem stage
    - Change in a single coefficient affects all poles in DF

• Preferred over DF because of numerical stability
Parallel (Partial Fraction) Form

- Decompose transfer function using a partial fraction expansion
  - \( H(z) = H_1(z) + H_2(z) + \ldots + H_K(z) \)
    - \( H_k(z) = \frac{b_{0k} + b_{1k}z^{-1}}{1 + a_{1k}z^{-1} + a_{2k}z^{-2}} \)
- Be sure to remember that PFE requires numerator order less than denominator
  - Use polynomial long division