6.50. Problem from Exam 1 Spring2001 Appears in: Fall02 PS4.

Problem

This is a problem from Exam 1 from Spring 2001. In this problem, we consider the implementation of a causal filter with system function

\[ H(z) = \frac{1}{(1 - 0.63z^{-1})(1 - 0.83z^{-1})} = \frac{1}{1 - 1.46z^{-1} + 0.5229z^{-2}} \]

This system is to be implemented with \((B + 1)\)-bit 2's-complement rounding arithmetic with products rounded before additions are performed. The input to the system is a zero-mean, white, wide-sense stationary random process, with values uniformly distributed between \(-S\) and \(S\), where \(S\) is a parameter.

1. Draw the direct form flow graph implementation for the filter, with all coefficient multipliers rounded to the nearest tenth.

2. Draw a flow graph implementation of this system as a cascade of two first-order systems, with all coefficient multipliers rounded to the nearest tenth.

3. Only one of the implementations from parts (1) and (2) above is usable. Which one? Explain.

4. To prevent overflow at the output node, we must carefully choose the parameter \(S\). For the implementation you selected in part (3), determine a value for \(S\) which guarantees the output will stay between \(-1\) and \(1\). (Ignore any potential overflow at nodes other than the output).

5. Redraw the flow graph you selected in part (3), this time including linearized noise models representing quantization roundoff error.

6. Whether you chose the direct form or cascade implementation for part (3), there is still at least one more design alternative:

   (a) If you chose the direct form, you could also use a transposed direct form.

   (b) If you chose the cascade form, you could implement the smaller pole first or the larger pole first.

For the system you chose in part (3), which alternative (if any) has lower output quantization noise power? Note you do not need to explicitly calculate the total output quantization noise power, but you must justify your answer with some analysis.
Solution from Fall02 PS4

(1) Direct Form Implementation:

(2) Cascade Implementation:

(3) Only the cascade implementation is usable. The quantized direct form implementation is not stable:

\[ H(z) = \frac{1}{1 - 1.5z^{-1} + .5z^{-2}} = \frac{1}{(1 - z^{-1})(1 - .5z^{-1})} \]

Pole at 1 on the unit circle, system is not stable b/c ROC does not include pole.

(4) \[ S < \frac{1}{\sum_{n=0}^{\infty} |h[n]|} \]

Solving by partial fractions we get:

\[ h[n] = 1.923((.6)^n u[n] + (.8)^n u[n]) \]

\[ \sum_{n=0}^{\infty} |h[n]| = 1.923(\sum_{n=0}^{\infty} (.6)^n + \sum_{n=0}^{\infty} (.8)^n u[n]) \]

Summation of an infinite geometric series:
\[ 1.923 \left( \frac{1}{1 - 0.6} + \frac{1}{1 - 0.8} \right) = 14.4225 \]

\[ S < \frac{1}{14.4225} = 0.69 \]

(5) Quantize after each multiply, add linear error sources there:

(6) We should put the large pole first in the cascade. The larger pole, at \( z = 0.8 \) is closer to the unit circle and thus amplifies the noise variance more at the output. Thus by putting it second the second noise source, gets amplified by a lesser amount by only going through the second pole. The first noise source will always go through the entire system, so we do not have control over that.

Solution from Spring01 exam

N/A
2. A sixth-order filter with system function

\[ H(z) = \frac{(1 + z^{-2})(1 + z^{-1})^2(1 - 2 \cos(\pi/6)z^{-1} + z^{-2})}{(1 - 1.6 \cos(\pi/4)z^{-1} + 0.64z^{-2})(1 + 1.6 \cos(\pi/4)z^{-1} + 0.64z^{-2})(1 - 1.8 \cos(\pi/4)z^{-1} + 0.81z^{-2})} \]

is to be implemented as a cascade of second-order sections. Considering only the effects of round-off noise, determine what is the best pole-zero pairing and the best ordering of the second-order sections.

Draw the pole/zero plot and give \( H_i(z) \) for each of the \( i = 1, 2, 3 \) stages.

**Solution**

First the pole/zero locations must be determined as shown in the following pole/zero plot.

Note: the location of complex conjugate root pairs result in factors of the form

\[ (1 - 2r \cos(\theta)z^{-1} + r^2 z^{-2}) \]

This results in zeros on the unit circle with two at \( z = -1 \), a pair at \( \pm j \), and a pair at \( z = e^{\pm j\pi/6} \). The pole pairs are at \( z = 0.8e^{\pm j\pi/4} \), \( z = 0.8e^{\pm j3\pi/4} \), and \( z = 0.9e^{\pm j\pi/6} \).

The stages are constructed by pairing zeros to the poles closest to the unit circle resulting in
the following 3 stages:

\[ H_1(z) = \frac{1 - 2 \cos(\pi/6) z^{-1} + z^{-2}}{1 - 1.8 \cos(\pi/4) z^{-1} + 0.81 z^{-2}} \]

\[ H_2(z) = \frac{1 + z^{-2}}{(1 - 1.6 \cos(\pi/4) z^{-1} + 0.64 z^{-2})} \]

\[ H_3(z) = \frac{(1 + z^{-1})^2}{(1 + 1.6 \cos(\pi/4) z^{-1} + 0.64 z^{-2})} \]

Notice that \( H_2(z) \) and \( H_3(z) \) were ordered arbitrarily since the poles are equidistant from the unit circle. Also, another eligible choice would be to reverse the stage ordering.
7.2. Recall that \( \Omega = \omega / T_d \).

(a) Then

\[
0.89125 \leq |H(j\Omega)| \leq 1, \quad 0 \leq |\Omega| \leq 0.2\pi / T_d
\]

\[
|H(j\Omega)| \leq 0.17783, \quad 0.3\pi / T_d \leq |\Omega| \leq \pi / T_d
\]

The plot of the tolerance scheme is

(b) As in the book's example, since the Butterworth frequency response is monotonic, we can solve

\[
|H_c(j0.2\pi / T_d)|^2 = \frac{1}{1 + \left(\frac{0.2\pi}{\Omega_c T_d}\right)^2N} = (0.89125)^2
\]

\[
|H_c(j0.3\pi / T_d)|^2 = \frac{1}{1 + \left(\frac{0.3\pi}{\Omega_c T_d}\right)^2N} = (0.17783)^2
\]

to get \( \Omega_c T_d = 0.70474 \) and \( N = 5.8858 \). Rounding up to \( N = 6 \) yields \( \Omega_c T_d = 0.7032 \) to meet the specifications.

(c) We see that the poles of the magnitude-squared function are again evenly distributed around a circle of radius 0.7032. Therefore, \( H_c(z) \) is formed from the left half-plane poles of the magnitude-squared function, and the result is the same for any value of \( T_d \). Correspondingly, \( H(z) \) does not depend on \( T_d \).
7.4. (a) In the impulse invariance design, the poles transform as \( z_k = e^{\pi T_d} \) and we have the relationship:

\[
\frac{1}{s + a} \leftrightarrow \frac{T_d}{1 - e^{-aT_d} s^{-1}}
\]

Therefore,

\[
H_c(s) = \frac{2/T_d}{s + 0.1} - \frac{1/T_d}{s + 0.2} = \frac{1}{s + 0.1} - \frac{0.5}{s + 0.2}
\]

The above solution is not unique due to the periodicity of \( z = e^{j\omega} \). A more general answer is

\[
H_c(s) = \frac{2/T_d}{s + (0.1 + j\frac{2\pi k}{T_d})} - \frac{1/T_d}{s + (0.2 + j\frac{2\pi l}{T_d})}
\]

where \( k \) and \( l \) are integers.

(b) Using the inverse relationship for the bilinear transform,

\[
z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}
\]

we get

\[
H_c(s) = \frac{2}{1 - e^{-0.2} \left( \frac{s + 1}{1 + s} \right)} - \frac{1}{1 - e^{-0.4} \left( \frac{s + 1}{s + 1 + e^{-0.4}} \right)}
\]

\[
= \frac{2(s + 1)}{s(1 + e^{-0.3}) + (1 - e^{-0.3})} - \frac{1}{s(1 + e^{-0.4}) + (1 - e^{-0.4})}
\]

\[
= \left( \frac{2}{s + 1 + e^{-0.2}} \right) - \left( \frac{1}{s + 1 + e^{-0.4}} \right)
\]

Since the bilinear transform does not introduce any ambiguity, the representation is unique.
7.5. (a) We must use the minimum specifications:

\[
\delta = 0.01 \\
\Delta \omega = 0.05\pi \\
A = -20 \log_{10}\delta = 40 \\
M + 1 = \frac{A - 8}{2.285\Delta \omega} + 1 = 90.2 \rightarrow 91 \\
\beta = 0.5842(A - 21)^{0.4} + 0.07886(A - 21) = 3.395
\]

(b) Since it is a linear phase filter with order 90, it has a delay of 90/2 = 45 samples.

(c) 

\[h_d[n] = \frac{\sin(0.625\pi(n - 45)) - \sin(0.3\pi(n - 45))}{\pi(n - 45)}\]
7.9. Using the relation $\omega = \Omega T$, the cutoff frequency $\omega_c$ for the resulting discrete-time filter is

\[
\omega_c &= \Omega_c T \\
&= [2\pi(1000)][0.0002] \\
&= 0.4\pi \text{ rad}
\]
7.10. Using the bilinear transform frequency mapping equation,

\[ \omega_c = 2 \tan^{-1} \left( \frac{\Omega_s T}{2} \right) \]

\[ = 2 \tan^{-1} \left( \frac{2\pi (2000)(0.4 \times 10^{-3})}{2} \right) \]

\[ = 0.7589 \pi \text{ rad} \]
7.13. Using the relation $\omega = \Omega T$,

$$
T = \frac{\omega_c}{\Omega_c} = \frac{2\pi/5}{2\pi(4000)} = 50 \text{ } \mu s
$$

This value of $T$ is unique. Although one can find other values of $T$ that will alias the continuous-time frequency $\Omega_c = 2\pi(4000) \text{ rad/s}$ to the discrete-time frequency $\omega_c = 2\pi/5 \text{ rad}$, the resulting aliased filter will not be the ideal lowpass filter.
7.14. Using the bilinear transform frequency mapping equation,

\[
\Omega_c = \frac{2}{T} \tan \left( \frac{\omega_c + 2\pi k}{2} \right), \quad k \text{ an integer}
\]

\[
= \frac{2}{T} \tan \left( \frac{\omega_c}{2} \right)
\]

\[
T = \frac{2}{2\pi(300)} \tan \left( \frac{3\pi/5}{2} \right) = 1.46 \text{ ms}
\]

The only ambiguity in the above is the periodicity in \( \omega \). However, the periodicity of the tangent function "cancels" the ambiguity and so \( T \) is unique.
7.22. A. Strictly speaking, the input $x(t)$ must be bandlimited to 5000 Hz to ensure that there is no aliasing when sampled at 10000 samples/sec. As a practical matter, it may be adequate to bandlimit the input to 7000 Hz. Frequency components between 5000 and 7000 Hz will alias to the range $\Omega = 2\pi 3000$ to $2\pi 5000$ rad/s, or $\omega = 0.6\pi$ to $\pi$, using $\omega = \Omega T$. Thus the aliased components will fall in the stopband of the discrete-time lowpass filter.

B. For the continuous-time system, the passband edge is $\Omega_p = \omega_p / T = 0.4\pi \times 10000 = 2\pi 2000$ rad/s. The stopband edge is $\Omega_s = \omega_s / T = 0.6\pi \times 10000 = 2\pi 3000$ rad/s. Within the passband the specifications are

$$1 - \delta_p \leq |H_{\text{eff}}(j\Omega)| \leq 1 + \delta_p, \quad |\Omega| \leq \Omega_p$$

$$0.99 \leq |H_{\text{eff}}(j\Omega)| \leq 1.02, \quad |\Omega| \leq 2\pi 2000.$$ 

Within the stopband the specifications are

$$|H_{\text{eff}}(j\Omega)| \leq \delta_s, \quad \Omega_s \leq \Omega \leq 2\pi 5000$$

$$|H_{\text{eff}}(j\Omega)| \leq 0.001, \quad 2\pi 3000 \leq \Omega \leq 2\pi 5000.$$ 

C. The given filter is a linear phase filter whose impulse response has a length of 28 samples. The group delay of the filter is $\alpha = 27/2 = 13.5$ samples. Since samples are spaced $10^{-4}$ seconds apart, the delay in seconds is $13.5 \times 10^{-4} = 1.35$ ms.