1 Fourier Series of Continuous Periodic Signals

Suppose $x(t)$ can be represented as a linear combination of harmonic complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{synthesis equation}$$

then the coefficients $\{a_k\}$ can be found as

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad \text{analysis equation}$$

where the a_k values are known as the Fourier Series coefficients or spectral coefficieints, ω_0 is the fundamental frequency and $T = \frac{2\pi}{\omega_0}$ is the fundamental period.

2 CTFS Pair Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{1}$$

multiply both sides by the complex exponential and integrate

$$x(t)e^{-jn\omega_0 t} = e^{-jn\omega_0 t}\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{2}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T e^{-jn\omega_0 t}\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}dt \tag{3}$$

$$= \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}dt \tag{4}$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt \tag{5}$$

Consider the integral on right side of eq (5)

$$\begin{aligned}
\int_0^T e^{j(k-n)\omega_0 t}dt &= \left[\frac{1}{j(k-n)\omega_0}e^{j(k-n)\omega_0 t}\right]_0^T \\
&= \frac{1}{j(k-n)\omega_0}\left[e^{j(k-n)\omega_0 T} - 1\right] \\
&= \frac{1}{j(k-n)\omega_0}\left[e^{j(k-n)\omega_0 \frac{2\pi}{\omega_0}} - 1\right]
\end{aligned}$$

let $m = k - n$

$$\begin{aligned}
&= \frac{1}{j(k-n)\omega_0}\left[e^{jm2\pi} - 1\right] \\
&= \frac{1}{j(k-n)\omega_0}[1-1] \\
&= 0 \qquad \text{for } k \neq n
\end{aligned}$$

When $k = n$

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T e^{j(0)\omega_0 t} dt = \int_0^T dt = T$$

Therefore

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases} \tag{6}$$

This demonstrates the orthognoality of harmonically related complex exponentials. That is, the inner product $\left\langle e^{jk\omega_0 t}, e^{jkn\omega_0 t} \right\rangle = 0$.

Returning to eq (5),

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$$

because only the $k = n$ term remains in eq (6)

$$= a_n T$$

and solving for the FS coefficients a_n

$$\Rightarrow a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$