Suppose $x(t)$ can be expressed as a linear combination of harmonic complex exponentials

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]  

**synthesis equation**

Then the FS coefficients \( \{a_k\} \) can be found as

\[ a_k = \frac{1}{T} \int_T^T x(t) e^{-jk\omega_0 t} dt \]  

**analysis equation**

- \( \omega_0 \) - fundamental frequency
- \( T = 2\pi/\omega_0 \) - fundamental period
- \( a_k \) known as FS coefficients or spectral coefficients
While we can prove this, it is not well suited for slides.

- See additional handout for details

Key observation from proof: Complex exponentials are orthogonal
VECTOR SPACE OF PERIODIC SIGNALS

All signals

Periodic signals, $\omega_0$
Each of the harmonic exponentials are orthogonal to each other and span the space of periodic signals.

The projection of $x(t)$ onto a particular harmonic ($a_k$) gives the contribution of that complex exponential to building $x(t)$.

- $a_k$ is how much of each harmonic is required to construct the periodic signal $x(t)$.
**HARMONICS**

- \( k = \pm 1 \Rightarrow \) fundamental component (first harmonic)
  - Frequency \( \omega_0 \), period \( T = 2\pi/\omega_0 \)
- \( k = \pm 2 \Rightarrow \) second harmonic
  - Frequency \( \omega_2 = 2\omega_0 \), period \( T_2 = T/2 \) (half period)
- ...
- \( k = \pm N \Rightarrow \) Nth harmonic
  - Frequency \( \omega_N = N\omega_0 \), period \( T_N = T/N \) (1/N period)
- \( k = 0 \Rightarrow a_0 = \frac{1}{T} \int_T x(t)dt \), DC, constant component, average over a single period
HOW TO FIND FS REPRESENTATION

- Will use important examples to demonstrate common techniques

- Sinusoidal signals – Euler’s relationship
- Direct FS integral evaluation
- FS properties table and transform pairs
\[ x(t) = 1 + \frac{1}{2} \cos 2\pi t + \sin 3\pi t \]

- First find the period
  - Constant 1 has arbitrary period
  - \( \cos 2\pi t \) has period \( T_1 = 1 \)
  - \( \sin 3\pi t \) has period \( T_2 = 2/3 \)
  - \( T = 2, \omega_0 = 2\pi/T = \pi \)

- Rewrite \( x(t) \) using Euler’s and read off \( a_k \) coefficients by inspection

\[ x(t) = 1 + \frac{1}{4} \left[ e^{j2\omega_0 t} + e^{-j2\omega_0 t} \right] + \frac{1}{2j} \left[ e^{j3\omega_0 t} - e^{-j3\omega_0 t} \right] \]

- Read off coeff. directly
  - \( a_0 = 1 \)
  - \( a_1 = a_{-1} = 0 \)
  - \( a_2 = a_{-2} = 1/4 \)
  - \( a_3 = 1/2j, a_{-3} = -1/2j \)
  - \( a_k = 0, \text{ else} \)
\[
x(t) = \begin{cases} 
1 & |t| < T_1 \\
0 & T_1 < |t| < T/2 
\end{cases}
\]

\[
k \neq 0 \quad a_k = \frac{1}{T} \int_{T} e^{-j k \omega_0 t} dt
\]

\[
= -\frac{1}{j k \omega_0 T} \left[ e^{-j k \omega_0 t} \right]_{-T_1}^{T_1} = \frac{1}{j k \omega_0 T} \left[ e^{j k \omega_0 T_1} - e^{-j k \omega_0 T_1} \right]
\]

\[
= \frac{2}{k \omega_0 T} \left[ \frac{e^{j k \omega_0 T_1} - e^{-j k \omega_0 T_1}}{2j} \right] = \frac{2 \sin (k \omega_0 T_1)}{k \omega_0 T}
\]

\[
= \frac{\sin (k \omega_0 T)}{k \pi} \cdot \text{modulated sin function}
\]

\[k = 0 \quad a_0 = \frac{1}{T} \int_{T} x(t) dt = \frac{1}{T} \int_{-T}^{T} dt = \frac{2T_1}{T}
\]

\[
x(t) = \begin{cases} 
1 & |t| < T_1 \\
0 & T_1 < |t| < T/2 
\end{cases} \quad \leftrightarrow \quad a_k = \begin{cases} 
\frac{2T_1/T}{\sin (k \omega_0 T)} & k = 0 \\
\frac{\sin (k \omega_0 T)}{k \pi} & k \neq 0
\end{cases}
\]
SINC FUNCTION

- Important signal/function in DSP and communication
  - sinc($x$) = $\frac{\sin \pi x}{\pi x}$ normalized
  - sinc($x$) = $\frac{\sin x}{x}$ unnormalized
- Modulated sine function
  - Amplitude follows 1/x
  - Must use L’Hopital’s rule to get $x=0$ time
• Consider different “duty cycle” for the rectangle wave
  • $T = 4T_1$ 50% (square wave)
  • $T = 8T_1$ 25%
  • $T = 16T_1$ 12.5%

• Note all plots are still a sinc shape
  • Difference is how the sync is sampled
  • Longer in time (larger $T$) smaller spacing in frequency $\rightarrow$ more samples between zero crossings

Figure 3.7 Plots of the scaled Fourier series coefficients $T_{nk}$ for the periodic square wave with $T_1$ fixed and for several values of $T$: (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega_{T_1} \omega_k)$, where the spacing between samples, $2\pi/T$, decreases as $T$ increases.
SQUARE WAVE

- Special case of rectangle wave with $T = 4T_1$
- One sample between zero-crossing

\[ a_k = \begin{cases} 
  1/2 & \text{if } k = 0 \\
  \frac{\sin(k\pi/2)}{k\pi} & \text{else}
\end{cases} \]
\[ x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \]

- Using FS integral

\[
a_k = \frac{1}{T} \int_{T} x(t)e^{-jk\omega_0 t} dt
\]

\[
= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt
\]

- Notice only one impulse in the interval

\[
= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt
\]

\[
a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0} dt = \frac{1}{T}
\]
PROPERTIES OF CTFS

- Since these are very similar between CT and DT, will save until after DT

- Note: As for LT and Z Transform, properties are used to avoid direct evaluation of FS integral
  - Be sure to bookmark properties in Table 3.1 on page 206