A Random Variable is a function that maps an event to a probability (real value)

Will use distribution functions to describe the functional mapping

Example: your score on the midterm is a random variable and the Gaussian distribution explains the probability you achieved a certain value (e.g. 70/100)
RANDOM VARIABLE

- $X(\xi)$ is a single-valued real function that assigns a real number (value) to each sample point (outcome) in a sample space $S$
  - Often just use $X$ for simplicity
  - This is a function (mapping) from sample space $S$ (domain of $X$) to values (range)
  - This is a many-to-one mapping
    - Different $\xi_i$ may have same value $X(\xi_i)$, but two values cannot come from same outcome
EVENTS DEFINED BY RVS

- Event
  - \((X = x) = \{\xi : X(\xi) = x\}\)
  - RV \(X\) value is \(x\), a fixed real number
- Similarly,
  - \((x_1 < X \leq x_2) = \{\xi : x_1 < X(\xi) \leq x_2\}\)
- Probability of event
  - \(P(X = x) = P\{\xi : X(\xi) = x\}\)
EXAMPLE: COIN TOSS 3 TIMES

- Sample space $S = \{HHH, HHT, ..., TTT\}$, $|S| = 2^3 = 8$
- Define RV $X$ as the number of heads after the three tosses
- Find $P(X = 2)$
  - Event A: $(X = 2) = \{\xi: X(\xi) = 2\} = \{HHT, HTH, HTT\}$
  - By equally likely events
    - $P(A) = P(X = 2) = \frac{|A|}{|S|} = \frac{3}{8}$
- Find $P(X < 2)$
  - Event B: $(X < 2) = \{\xi: X(\xi) < 2\} = \{HTT, THH, HTT, TTT\}$ (1 or less heads)
  - By equally likely events
    - $P(B) = P(X < 2) = \frac{|B|}{|S|} = \frac{4}{8} = \frac{1}{2}$
CUMULATIVE DISTRIBUTION FUNCTION (CDF)

- $F_x(x) = P(X \leq x)$  \(-\infty < x < \infty\)
  - $F$ – the CDF
  - $X$ – the RV of interest
  - $x$ – the value the RV will take

- Note: this is an increasing (non-decreasing) function
CDF PROPERTIES

1) \( 0 \leq F_X(x) \leq 1 \)
   - Must be less than some maximal value

2) \( F_X(x_1) \leq F_X(x_2) \) if \( x_1 < x_2 \)
   - Non-decreasing function

...  

5) \[ \lim_{x \to a^+} F_X(x) = F_X(a^+) = F_X(x) \]
with \( a^+ = \lim_{0<\epsilon\to0} a + \epsilon \)
   - Continuous from the right
**EXAMPLE: 3 COIN TOSS AGAIN**

- $X$ – number of heads in three tosses

<table>
<thead>
<tr>
<th>$x$ (value)</th>
<th>Event ($X \leq x$)</th>
<th># elements</th>
<th>$F_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>$\emptyset$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>${\text{TTT}}$</td>
<td>1 $(1 + 0)$</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>3</td>
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- $X$ – number of heads in three tosses

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<tr>
<td>1</td>
<td>${HTT, THT, TTH, TTT}$</td>
<td>4 ($3 + 1$)</td>
<td>$\frac{4}{8} = \frac{1}{2}$</td>
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<tr>
<td>2</td>
<td>{HHT, HTH, THH, HTT, THT, TTH, TTT}</td>
<td>7 (3 + 4)</td>
<td>$\frac{7}{8}$</td>
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<td>${\text{HHT, HTH, THH, HTT, THT, TTH, TTT}}$</td>
<td>7 (3 + 4)</td>
<td>$\frac{7}{8}$</td>
</tr>
<tr>
<td>3</td>
<td>${\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}}$</td>
<td>8 (1 + 7)</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$S$</td>
<td>8 (0 + 8)</td>
<td>1</td>
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### Example: 3 Coin Toss Again

- $X$ – number of heads in three tosses

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<tr>
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<td>{HTT, THT, TTH, TTT}</td>
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<td>$\frac{4}{8} = \frac{1}{2}$</td>
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<tr>
<td>2</td>
<td>{HHT, HTH, THH, HTT, THT, TTH, TTT}</td>
<td>7 (3 + 4)</td>
<td>$\frac{7}{8}$</td>
</tr>
<tr>
<td>3</td>
<td>{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}</td>
<td>8 (1 + 7)</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$S$</td>
<td>8 (0 + 8)</td>
<td>1</td>
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PROBABILITIES FROM CDF

- Completely specify probabilities from a CDF

1) \( P(a < X \leq b) = F_X(b) - F_X(a) \)
   \[= P(X \leq b) - P(X \leq a) \]

2) \( P(X > a) = 1 - F_X(a) \)

3) \( P(X < b) = F_X(b^-) \)
   \[b^- = \lim_{\epsilon \to 0} b - \epsilon \text{ for } 0 < \epsilon \to 0 \]

- Approach from the left side
• $X$ is RV with CDF $F_X(x)$ and $F_X(x)$ only changes in jumps (countably many) and is constant between jumps

• Range of $X$ contains a finite (countably infinite) number of points
### PROBABILITY MASS FUNCTION (PMF)

- Given jumps in discrete RV @ points $x_1, x_2, \ldots$ and $x_i < x_j$ for $i < j$
  
  - $p_x(x) = F_x(x_i) - F_x(x_{i-1})$
  
  - $= P(X \leq x_i) - P(X \leq x_{i-1}) = P(X = x_i)$

- 3 Coin toss example

<table>
<thead>
<tr>
<th>$x$ (value)</th>
<th># elements</th>
<th>$F_x(x)$</th>
<th>$p_x(x)$</th>
<th>Discussion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4 (3+1)</td>
<td>$\frac{4}{8} = \frac{1}{2}$</td>
<td>$p_x(1) = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$</td>
<td>&lt;how much more needed from previous value&gt;</td>
</tr>
<tr>
<td>2</td>
<td>7 (3 + 4)</td>
<td>$\frac{7}{8}$</td>
<td>$p_x(2) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$</td>
<td>3 extra outcomes</td>
</tr>
<tr>
<td>3</td>
<td>8 (1 + 7)</td>
<td>1</td>
<td>$p_x(3) = 1 - \frac{7}{8} = \frac{1}{8}$</td>
<td>1 extra outcome</td>
</tr>
</tbody>
</table>
PMF PROPERTIES

1) \( 0 \leq p_{X}(x_k) \leq 1 \quad k = 1, 2, \ldots \) (finite set of values)

2) \( p_{X}(x) = 0 \) if \( x \neq x_k \) (a value that cannot occur)

3) \( \Sigma_k p_{X}(x_k) = 1 \)

CDF from PMF

\[ F_X(x) = P(X \leq x) = \Sigma_{x_k \leq x} p_{X}(x_k) \]

Accumulation of probability mass
CONTINUOUS RV

- $X$ is RV with CDF $F_X(x)$ continuous and has a derivative $\frac{dF_X(x)}{dx}$ exists
- Range contains an interval of real numbers

- Note: $P(X = x) = 0$
  - There is zero probability for a particular continuous outcome $\Rightarrow$ only over a range of values
PROBABILITY DENSITY FUNCTION (PDF)

- \( f_X(x) = \frac{dF_X(x)}{dx} \) \( \quad \) pdf of \( X \)

Properties

1) \( f_X(x) \geq 0 \)

2) \( \int_{-\infty}^{\infty} f_X(x)dx = 1 \)

3) \( f_X(x) \) is piecewise continuous

4) \( P(a < X \leq b) = \int_{a}^{b} f_X(x)dx \)
   \( = P(a \leq X \leq b) \)
   \( = F_X(b) - F_X(a) \)

CDF from PDF

- \( F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(\xi)d\xi \)
MEAN

- Expected value of RV $X$
- Discrete
  - $\mu_X = E[X] = \sum_k x_k p_X(x_k)$
- Continuous
  - $\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$
MOMENT

- $n^{th}$ moment defined as
- Discrete
  - $E[X^n] = \sum_k x_k^n P_X(x_k)$
- Continuous
  - $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$
VARIANCE

- $\sigma_X^2 = Var(X) = E[(X - E[X])^2]$
  - $E[.]$ – expected value operation
  - $E[X] = \mu_X$ - mean
- Discrete
  - $\sigma_X^2 = \sum_k (x - \mu_X)^2 p_X(x_k)$
- Continuous
  - $\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$

$Var(X) = E[(X - E[X])^2]$

- $E[X^2] - 2\mu_X E[X] + \mu_X^2$
- $E[X^2] - 2\mu_X^2 + \mu_X^2$
- $E[X^2] - \mu_X^2$
- $\underbrace{E[X^2]}_{2nd\ moment} - \underbrace{E[X]}_{1st\ moment}$
IMPORTANT DISTRIBUTIONS

- Model real-world phenomena
- Mathematically convenient specification for probability distribution (usually pmf or pdf)

- Will examine similar discrete and continuous distributions
  - Note: will leave most of content for the book rather than in slides
BERNOULLI DISTRIBUTION

- Binary RV with probability $p$ of 1 ("success") or $(1 - p)$ for failure
  - E.g. a coin flip with heads a "success" or "1" and tails a "failure" or "0"

- $p_X(k) = P(X = k) = p^k (1 - p)^{1-k}$
  - $0 < p < 1$ is probability of success
  - $(1 - p)$ is probability of failure
  - $k = 0, 1$
RV to count the number of successes with $n$ independent Bernoulli trials

$p_X(k) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$

$n \choose k = \frac{n!}{k!(n-k)!}$ - $n$ choose $k$

Number of ways to get $k$ successes (heads) in $n$ trials (coin tosses)
Remember $P(A|B) = \frac{P(A \cap B)}{P(B)}$, $P(B) > 0$

- Conditional CDF
  - $F_X(x|B) = P(X \leq x|B) = \frac{P\{(X \leq x) \cap B\}}{P(B)}$

- Conditional PMF
  - $p_X(x_k|B) = P(X = x_k|B) = \frac{P\{(X = x_k) \cap B\}}{P(B)}$

- Conditional PDF
  - $f_X(x|B) = \frac{d}{dx} F_X(x|B)$