EE361: SIGNALS AND SYSTEMS II

CH5: RANDOM PROCESSES

NOTE ON CONTENT

- We will only cover 5.1-5.4.C
- Lots of other great content will be skipped (such as types of Random Processes)

INTRODUCTION

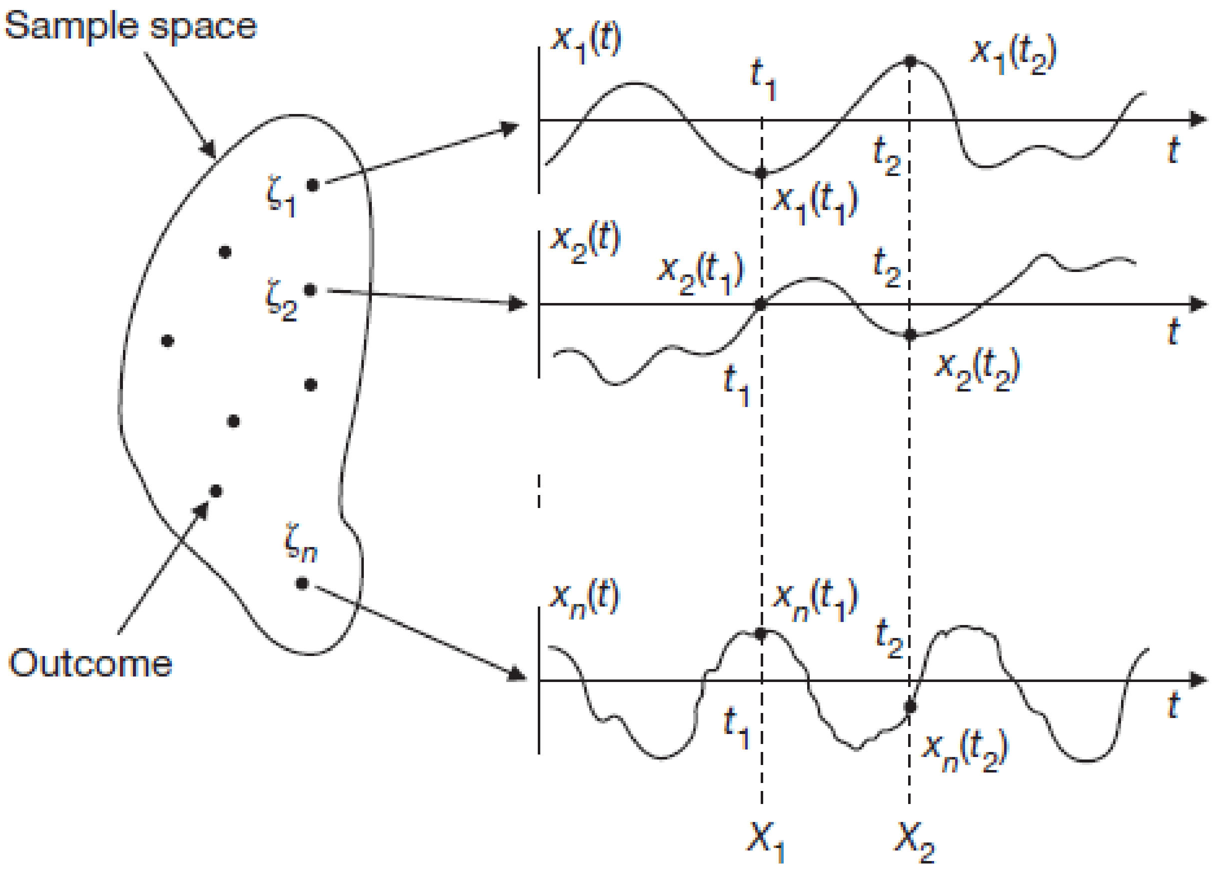
- Random signal – signals that take a random value at any given time and must be characterized statistically
  - E.g. noise in a physical system (static on microphone)
- When observing a random signal over time, there may be regularities that can be described using a probabilistic model
  - → Random process

RANDOM (STOCHASTIC) PROCESS DEFINITION

- A RP is a family of RVs $\{X(t), t \in T\}$ defined on a probability space, indexed by parameter t, where t varies over index set T
- Note: since a RV is a function defined over a sample space S, a RP is really a function of two arguments $\{X(t, \xi), t \in T, \xi \in S\}$

RP DEFINITION

- RP: $\{X(t,\xi), t \in T, \xi \in S\}$
- Fixed $t = t_k$
  - $X(t_k,\xi) = X_k(\xi) = X$
    - Random variable (depends on $\xi \in S$)
- Fixed $\xi = \xi_i \in S$
  - $X(t,\xi_i) = X_i(t)$
    - Single function of time t
    - Sample function or realization of process
  - All sample functions (totality of all realizations) is known as an ensemble
- Fixed $t = t_k$ and $\xi = \xi_i$
  - $X(t_k,\xi_i)$ is a real number



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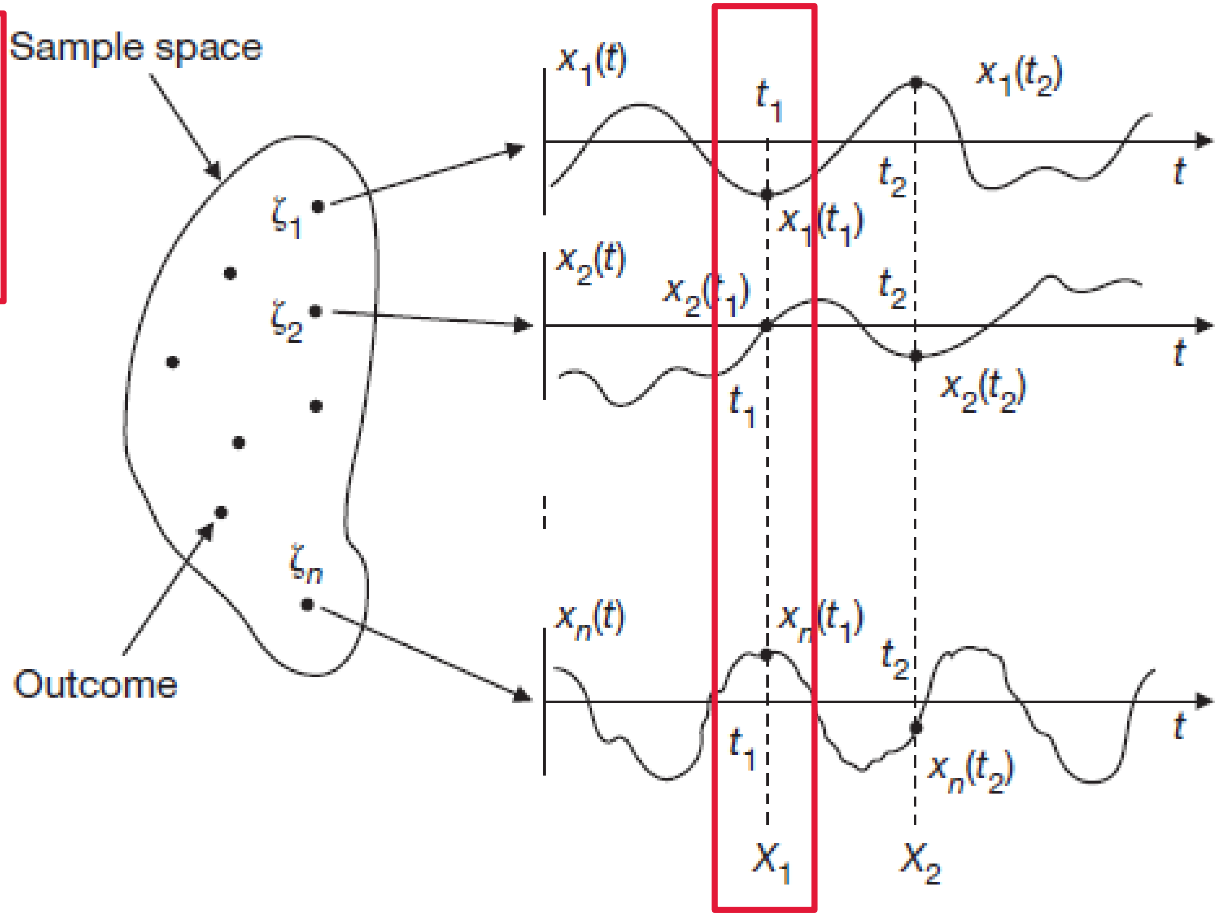

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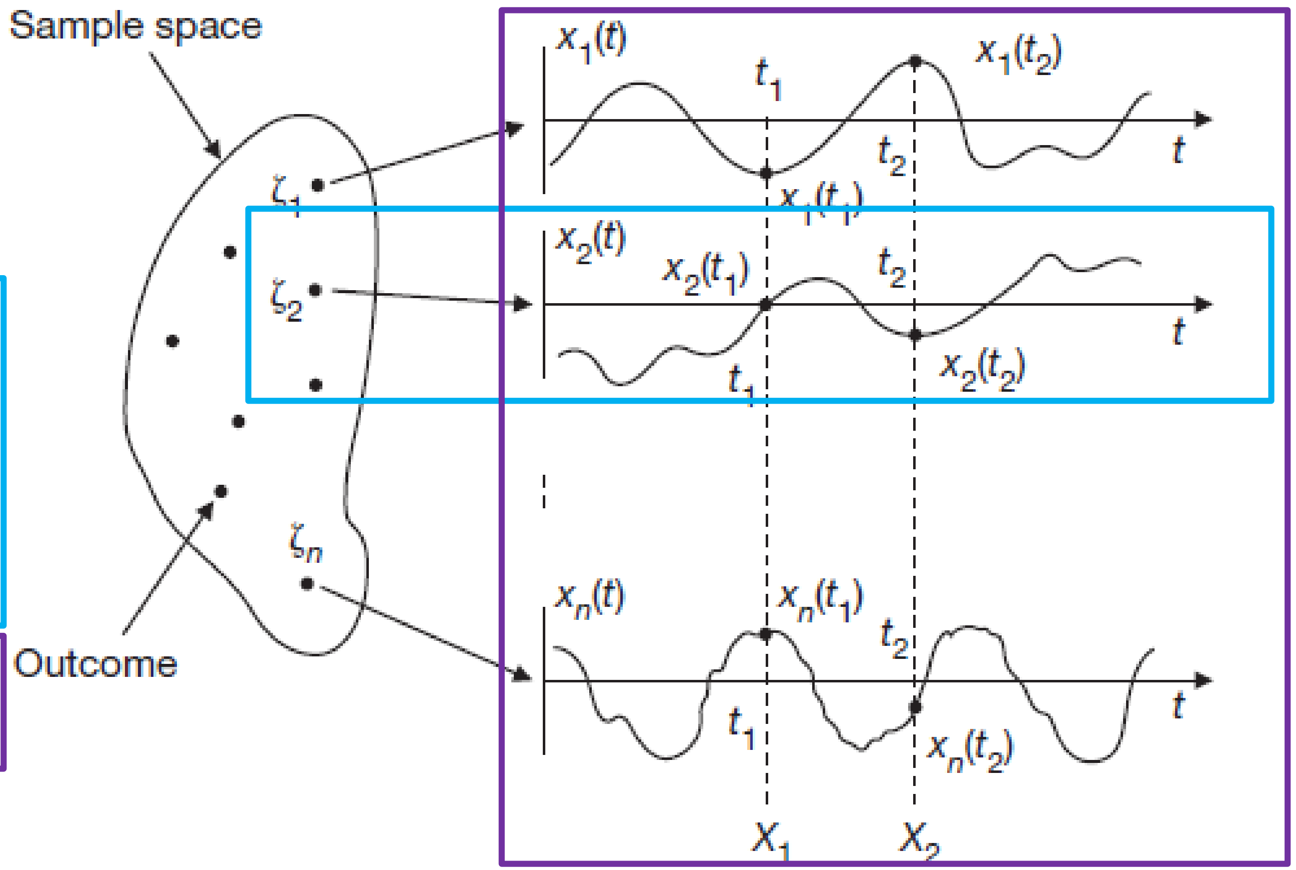

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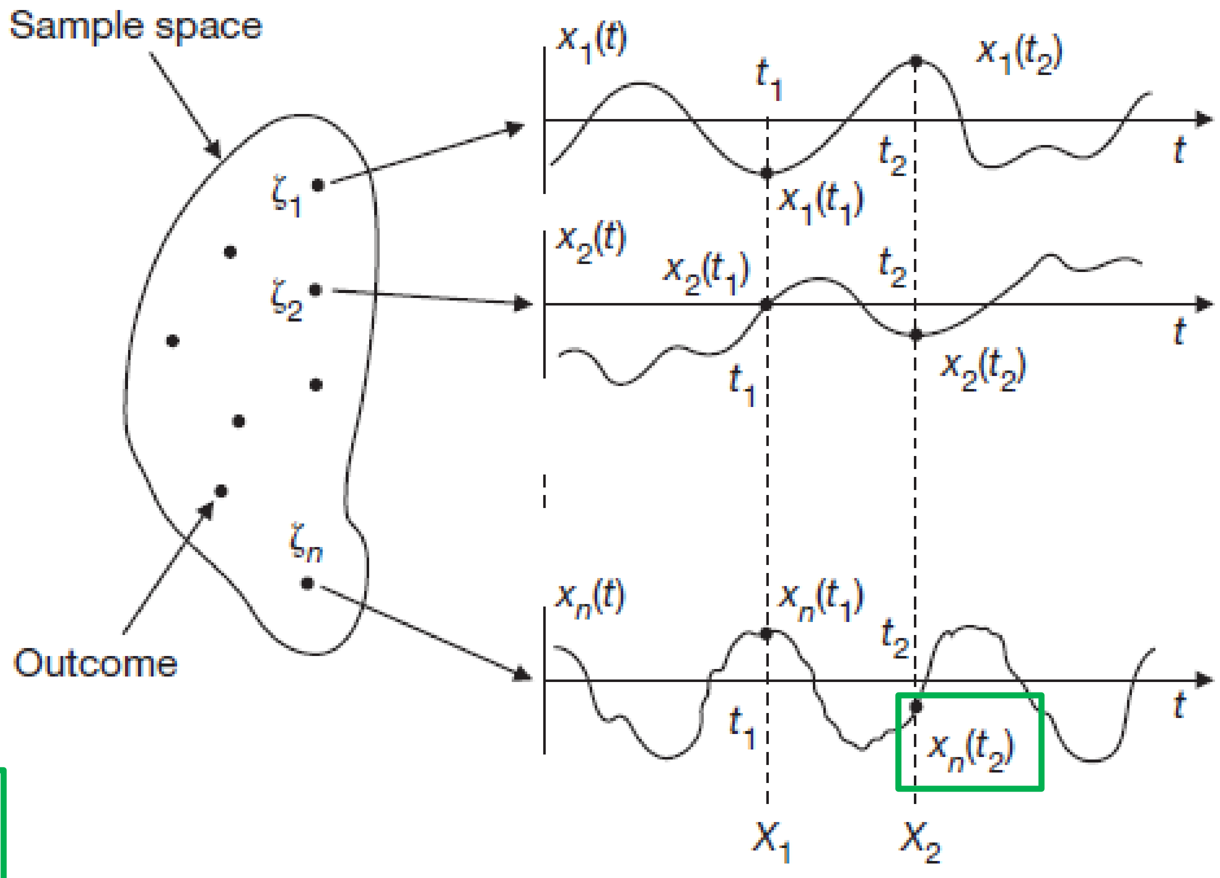

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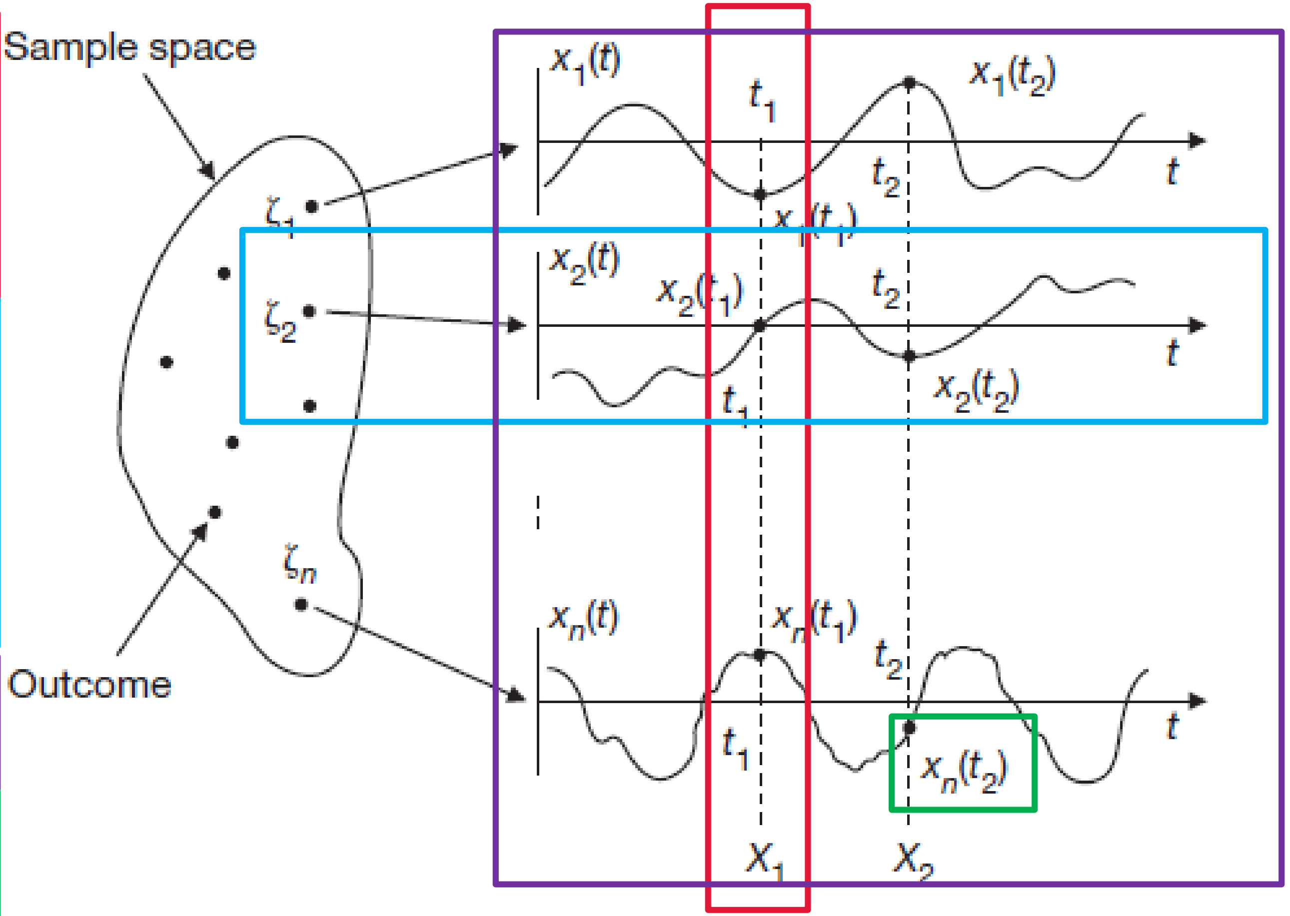
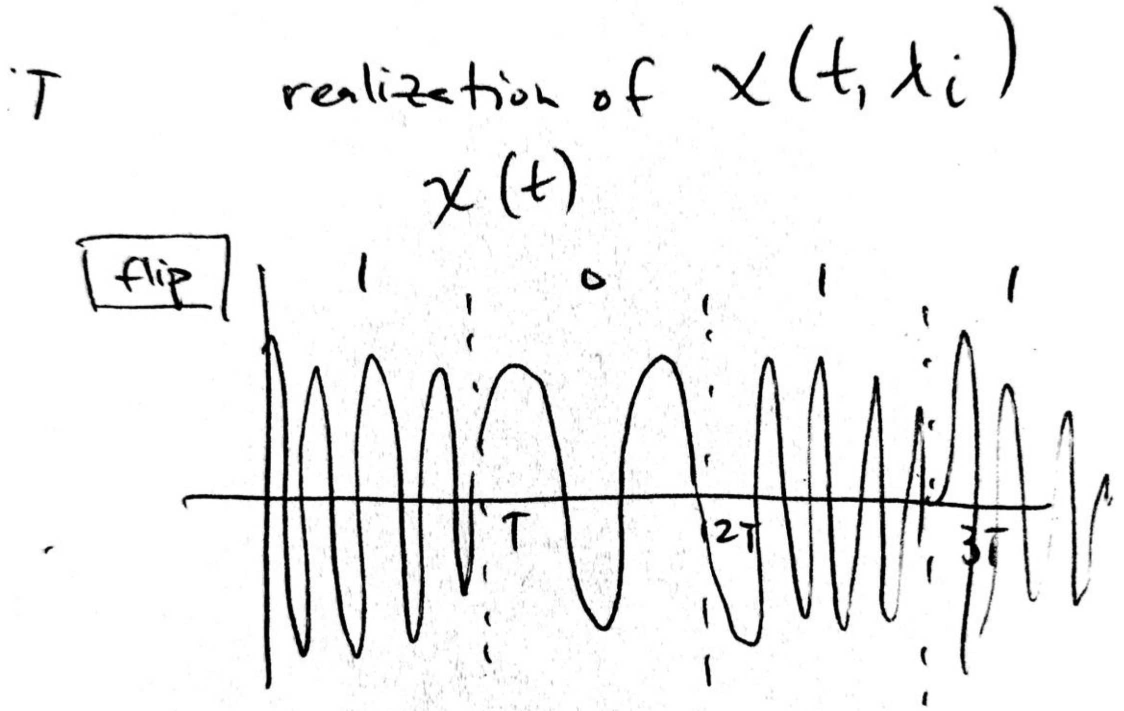


EXAMPLE 1

- Random coin flip experiment
  $S = \{H, T\}$
  - $X(t, H) = x_1(t) = \sin \omega_1 t$
  - $X(t, T) = x_2(t) = \sin \omega_2 t$
    - ω_1, ω_2 fixed numbers
- $X(t)$ is a random signal with $x_1(t), x_2(t)$ as sample functions
- Note: $x_1(t), x_2(t)$ are deterministic, randomness comes from the coin flip

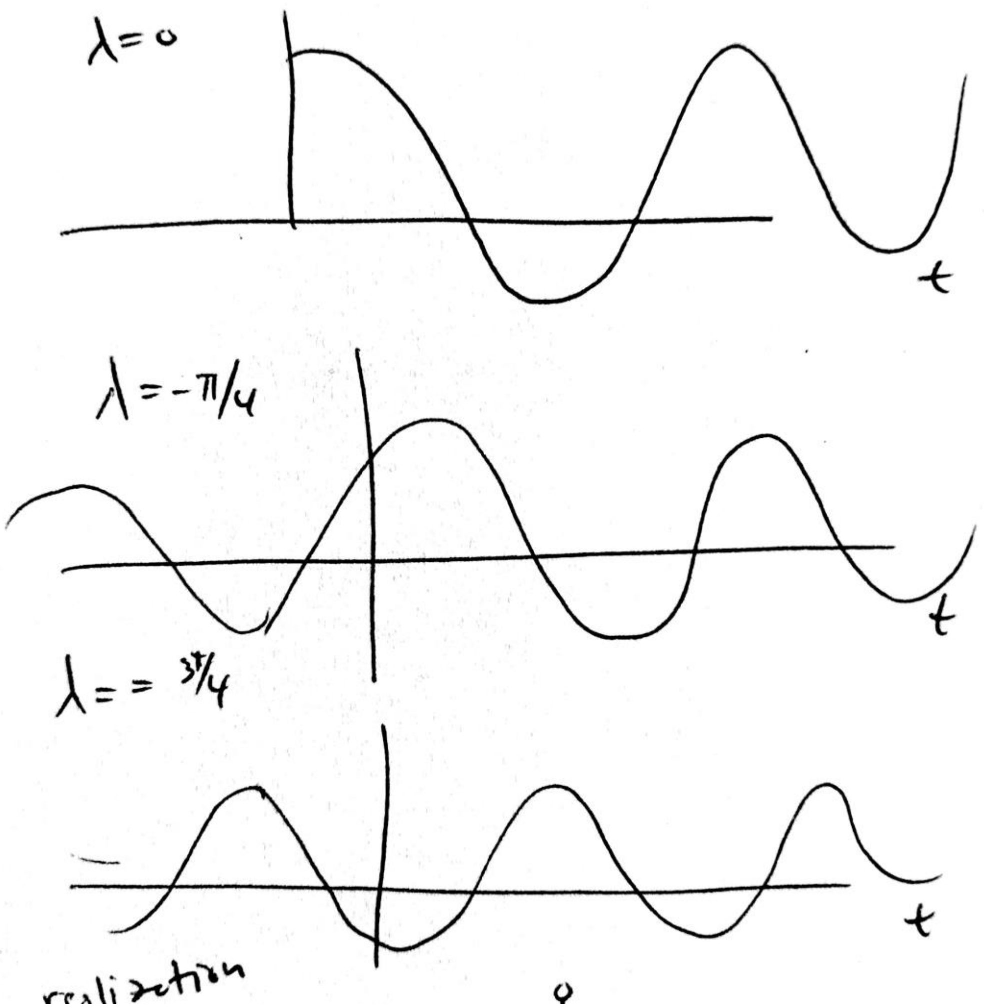
EXAMPLE 2

- Flip a coin repeatedly and observe sequence of outcomes
  - $S = \{\lambda_i, i = 1,2, \ldots\}$ where λ_i = H or T
- Let
  - $X(t, \lambda_i) = \sin(\Omega_i\, t)$
    - $(i-1)T \le t \le iT$
    - $\Omega_i = \begin{cases} \omega_1 & \lambda_i = H \\ \omega_2 & \lambda_i = T \end{cases}$
- The signal $X(t)$ is composed of sections of sinusoids with different frequency
  - E.g. flips are bits to be sent in comm



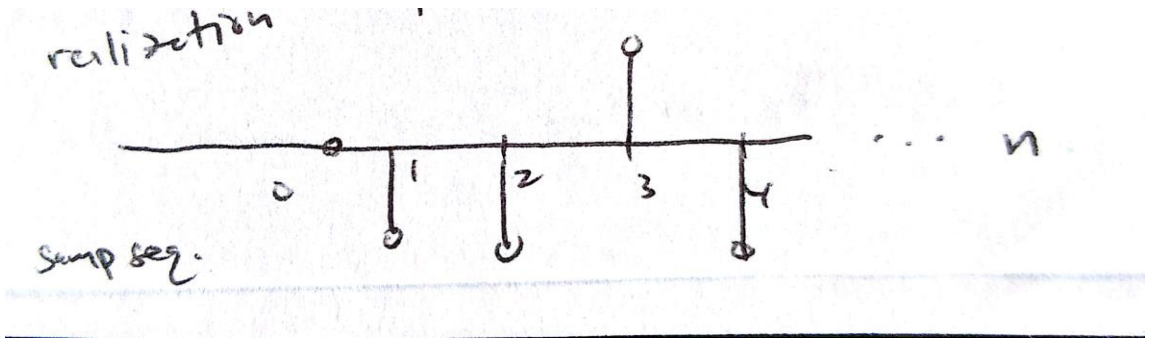
EXAMPLE 3

- Random signal defined with a RV
- $X(t) = a\cos(\omega_0 t + \Theta)$
  - $\Theta \sim U[0,2\pi]$
  - $\Theta(\xi) = \lambda \quad \forall \xi \in S = [0,2\pi]$
- $X(t, \xi) = a\cos(\omega_0 t + \lambda)$
  - $0 \leq \lambda \leq 2\pi$
  - Ensemble of cosines with same amplitude a and frequency ω_0 but with different phase λ



EXAMPLE 4

- $X_1, X_2, \dots$ independent RV with
- $P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2}$
- Let
  - $X(n) = \{X_n, n \geq 0\}$
  - $X_0 = 0$



- Define a DT random sequence

DESCRIPTION OF RANDOM PROCESS

- Given RP $\{X(t), t \in T\}$
  - T – parameter set
  - $X(t)$ values – states
    - All possible values/states make up the state space E
  - E – state space
- Discrete T set – discrete-parameter (discrete-time) process or random sequence
  - $\{X_n, n = 1, 2, \dots\}$
- Continuous T – continuous-parameter (continuous-time) process
- Discrete E – discrete-state processes (also called a chain)
  - $E = \{0, 1, 2, \dots\}$
- Continuous E – continuous-state process
- Complex RP
  - $X(t) = X_1(t) + jX_2(t)$
    - $X_1(t), X_2(t)$ are real random processes

CHARACTERIZATION OF RANDOM PROCESS

- Probabilistic description
  - [Difficult] Requires full knowledge of all distributions (like n-variate RV)
- Expectation-based statistics description
  - [Easier] Lower-order relationships which are common for many problems

PROBABILISTIC DESCRIPTION OF RP I

- Consider RP $X(t)$. For fixed time t_1, $X(t_1) = X_1$ is a RV with CDF
  - $F_X(x_1; t_1) = P(X(t_1) \leq x_1\}$
    - First-order distribution of $X(t)$
- Second-order distribution given t_1 and t_2
  - $F_X(x_1, x_2; t_1, t_2) = P(X(t_1) \leq x_1, X(t_2) \leq x_2)$
- Can easily generalize to nth-order distribution

PROBABILISTIC DESCRIPTION OF RP II

- Use of PDF to specify RP
- Discrete
  - $p_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{X(t_1) = x_1, \ldots, X(t_n) = x_n\}$
- Continuous
  - $f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{\partial F_X(x_1, \ldots, x_n; t_1, \ldots, t_n)}{\partial x_1 \ldots \partial x_n}$

STATISTICS OF RP I

- RP often described by statistical averages

- Mean $X(t)$ – ensemble average
  - $\mu_X(t) = E[X(t)]$
  - Note μ_X is a function of time and $X(t)$ is a RV for a fixed t
- Autocorrelation – describes relationship between two samples (@ 2 times) of $X(t)$
  - $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = R_{XX}(t_2, t_1)$
- Often expressed in terms of how far apart samples are in time
  - $R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$

STATISTICS OF RP II

- Autocovariance
  - $C_{XX}(t_1, t_2) = E\left[\left(X(t_1) - \mu_X(t_1)\right)\left(X(t_2) - \mu_X(t_2)\right)\right]$
    $= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$
- Note: both R_{XX} and C_{XX} are deterministic functions of t_1, t_2
  - $Var(X) = \sigma_X^2(t) = E\left[\left(X(t) - \mu_X(t)\right)^2\right] = C_{XX}(t, t)$

STATISTICS OF RP III

- Given random signals $X(t), Y(t)$
- Cross correlation
  - $R_{XY}(t_1, t_2) = E[X(t_1)X(t_2)]$
- Cross covariance
  - $C_{XY}(t_1, t_2) = E\big[\big(X(t_1) - \mu_X(t_1)\big)\big(Y(t_2) - \mu_Y(t_2)\big)\big]$
    $= R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2)$

EXAMPLE 5.12

- Consider RP
  - $X(t) = Y\cos\omega t$
- Where ω is a constant and $Y \sim U[0,1]$
- (a) Find $E[X(t)]$
- (b) Find autocorrelation $R_{XX}(t,s)$ of $X(t)$
- (c) Find autocovariance $C_{XX}(t,s)$

CLASSIFICATION OF RP

- Special RP can be specified just by first- and second-order distributions
  - Special probabilistic structure requires less information to specify
- Will consider processes that are:
  - Strict sense stationary
  - Wide sense stationary
  - Independent

(STRICT-SENSE) STATIONARY I

- RP $\{X(t), t \in T\}$ where for all n and every set of time instants $\{t_i \in T, i = 1, 2, \dots, n\}$
  - $F_X(x_1, \dots, x_n; t_1, \dots, t_n) = F_X(x_1, \dots, x_n; t_1 + \tau, \dots, t_n + \tau)$
- The distribution of a stationary process is unaffected by a shift (τ) in the time origin
  - $X(t)$ and $X(t + \tau)$ will have the same distribution
- Non-stationary processes have distributions that depend on time $t_1, \dots, t_n$

(STRICT-SENSE) STATIONARY II

- 1st-order distribution
  - $F_X(x;t) = F_X(x;t+\tau) = F_X(x)$
  - $f_X(x;t) = f_X(x)$
    - $\Rightarrow \mu_X(t) = E[X(t)] = \mu$ constant
    - $\Rightarrow Var\big(X(t)\big) = \sigma^2$ constant
- 2nd-order distribution
  - $F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_2 - t_1)$
  - $f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1)$
    - Characterized by time difference

WIDE SENSE STATIONARY

- Process that does not have stationary conditions $\forall n$ but for $n = 2$ (compare two times rather than all)
  - $E[X(t)] = \mu$ constant
  - $R_X(t, s) = E[X(t)X(s)] = R_X(|s - t|) = R_X(\tau)$
    - WSS process only depends on time difference τ
- Note: Avg. power of process is independent of t
  - $E[X^2(t)] = R_{XX}(0)$
- Similarly, jointly WSS
  - $R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)] = R_{XY}(\tau)$

INDEPENDENT PROCESSES

- Given RP $X(t)$, the RVs for fixed time t_i are independent

$$F_X(x_1, \dots, x_n; t_1, \dots, t_n) = \prod_{i=1}^{n} F_X(x_i; t_i)$$

  - Only first-order distributions are required

EXAMPLE 5.20

- Consider RP
  - $X(t) = A\cos(\omega t + \Theta)$
- Where A, ω is a constant and $\Theta \sim U[-\pi, \pi]$

- Show $X(t)$ is WSS
  - Check conditions:
  - 1) $E[X(t)] = \mu$ constant
  - 2) $R_{XX}(t, s) = R_{XX}(\tau)$ only depends on time difference