EE361: SIGNALS AND SYSTEMS II

CH4: FUNCTIONS OF RVS, EXPECTATION

INTRODUCTION

- We will only give a brief glimpse of a few concepts
- Would require much more time to do this properly

FUNCTION OF ONE RANDOM VARIABLE

- Define a new RV as function of another
  - $Y = g(X)$
- Denote D_Y as the subset of R_X (range of X) such that $g(X) \leq y$
  - $(Y \leq y) = [g(X) \leq y] = (X \in D_Y)$
  - Event of all outcomes ξ s.t. $X(\xi) \in D_Y$

DISTRIBUTIONS

- CDF
- $F_Y(y) = P(Y \leq y)$
  $= P[g(X) \leq y]$
  $= P(X \in D_Y)$
  $= \int_{D_Y} f_X(x)dx$
- Note: we will only consider continuous time

- PDF
- With X continuous with pdf $f_X(x)$ and $y = g(x)$ a one-to-one mapping
  - $x = g^{-1}(y) = h(y)$
- Then,
  - $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$
    $= f_X[h(y)]\left|\frac{dh(y)}{dy}\right|$

EXPECTATION (4.5 A, C)

$$E[Y] = E_X[g(X)] = \begin{cases} \sum_i g(x_i) p_X(x_i) & \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f_X(x) dx & \text{continuous} \end{cases}$$

- Note subscript X in $E_X[.]$ indicates the underlying randomness → distribution comes from RV X

- Linearity property
  - $E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i]$
- Independence property
  - $E[\prod_{i=1}^{n} g_i (X_i)] = \prod_{i=1}^{n} E[g_i(X_i)]$

SPR 4.92 EXPECTATION

- X number of heads in 3 tosses of a coin. Find expected value of $Y = X^2$

$$X \sim Bin(3, 0.5)$$
$$\Rightarrow p_X(k) = \binom{3}{k} 0.5^k 0.5^{3-k}$$
$$p_X(0) = 1/8$$
$$p_X(1) = 3/8$$
$$p_X(2) = 3/8$$
$$p_X(3) = 1/8$$

$$\begin{aligned} E[Y] &= E_X[Y] \\ &= E_X[X^2] = \sum_k x_k^2 p_X(x_k) \\ &= \sum_{k=0}^{3} k^2 p_X(k) \\ &= 0^2 p_X(0) + 1^2 p_X(1) + 2^2 p_X(x) + 3^2 p_X(3) \\ &= 1\left(\frac{3}{8}\right) + 4\left(\frac{3}{8}\right) + 9\left(\frac{1}{8}\right) \\ &= \frac{24}{8} = 3 \end{aligned}$$

SPR 4.87 DISTRIBUTION

- Find the pdf of Y if $X \sim U[-1, 2]$
  - $Y = 2X + 3 = g(X)$
- Find mapping and partial derivatives
  - $x = h(y) = \frac{y-3}{2}$ $\quad dy = 2dx \Rightarrow \frac{dx}{dy} = \frac{1}{2} = \frac{d}{dy}h(y)$
- Combine for final distribution
  - $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X(h(y))\left|\frac{dh(y)}{dy}\right| = f_X\left(\frac{y-3}{2}\right)\frac{1}{2} = \frac{1}{6}$
  - Note: $f_X(x) = \frac{1}{3}$ since it is a uniform RV
  - Check endpoints
    - $-1 = \frac{y-3}{2} \Rightarrow y = 3 - 2 = 1$ $\quad 2 = \frac{y-3}{2} \Rightarrow y = 7$

$f_Y(y) \sim U[1,7]$

- See problems 4.2 - 4.4