EE361: SIGNALS AND SYSTEMS II

CH2: RANDOM VARIABLES

http://www.ee.unlv.edu/~b1morris/ee361

RANDOM VARIABLES

CHAPTER 2.1-2.2

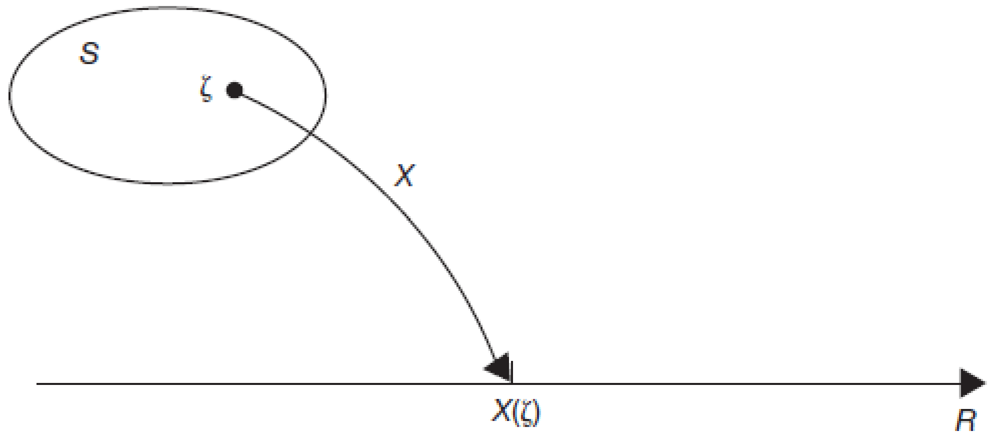
INTRODUCTION

- A Random Variable is a function that maps outcomes to values and allows for assignment of a probability (real value) to an event

- Will use distribution functions to describe the functional mapping
  - Example: your score on the midterm is a random variable and the Gaussian distribution explains the probability you achieved a certain value (e.g. 70/100)

RANDOM VARIABLE

- $X(\xi)$ is a single-valued real function that assigns a real number (value) to each sample point (outcome) in a sample space S
  - Often just use X for simplicity
  - This is a function (mapping) from sample space S (domain of X) to values (range)
  - This is a many-to-one mapping
    - Different ξ_i may have same value $X(\xi_i)$, but two values cannot come from same outcome



EVENTS DEFINED BY RVS

- Event
  - $(X = x) = \{\xi : X(\xi) = x\}$
  - RV X value is x, a fixed real number
- Similarly,
  - $(x_1 < X \leq x_2) = \{\xi : x_1 < X(\xi) \leq x_2\}$
- Probability of event
  - $P(X = x) = P\{\xi : X(\xi) = x\}$

EXAMPLE: COIN TOSS 3 TIMES

- Sample space $S = \{HHH, HHT, \dots, TTT\}$, $|S| = 2^3 = 8$
- Define RV X as the number of heads after the three tosses
- Find $P(X = 2)$
  - Event A: $(X = 2) = \{\xi : X(\xi) = 2\} = \{\text{HHT}, \text{HTH}, \text{THH}\}$
  - By equally likely events
    - $P(A) = P(X = 2) = \frac{|A|}{|S|} = \frac{3}{8}$
- Find $P(X < 2)$
  - Event B: $(X < 2) = \{\xi : X(\xi) < 2\} = \{\text{HTT}, \text{THT}, \text{HTT}, \text{TTT}\}$ (1 or less heads)
  - By equally likely events
    - $P(B) = P(X < 2) = \frac{|B|}{|S|} = \frac{4}{8} = \frac{1}{2}$

DISTRIBUTION FUNCTIONS

CHAPTER 2.3

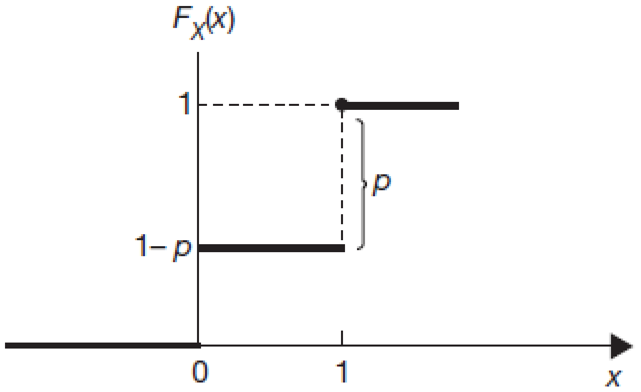
CUMULATIVE DISTRIBUTION FUNCTION (CDF)

- $F_X(x) = P(X \leq x) \qquad -\infty < x < \infty$
  - F – the CDF
  - X – the RV of interest
  - x – the value the RV will take

- Note: this is an increasing (non-decreasing) function

CDF PROPERTIES

- 1) $0 \leq F_X(x) \leq 1$
  - Must be less than some maximal value
- 2) $F_X(x_1) \leq F_X(x_2) \quad$ if $x_1 < x_2$
  - Non-decreasing function
- …
- 5) $\lim_{x \to a^+} F_X(x) = F_X(a^+) = F_X(x)$
  with $a^+ = \lim_{0 < \epsilon \to 0} a + \epsilon$
  - Continuous from the right



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EXAMPLE: 3 COIN TOSS AGAIN

- X – number of heads in three tosses

x (value)	Event ($X \leq x$)	# elements	$F_X(x)$
-1	∅	0	0
0	{TTT}	1 (1 + 0)	$\frac{1}{8}$
1			
2			
3			
4			

EXAMPLE: 3 COIN TOSS AGAIN

- X – number of heads in three tosses

x (value)	Event $(X \leq x)$	# elements	$F_X(x)$
-1	∅	0	0
0	{TTT}	**1** (1 + 0)	$\frac{1}{8}$
1	{HTT, THT, TTH, **TTT**}	4 (3+**1**)	$\frac{4}{8}=\frac{1}{2}$
2			
3			
4			

EXAMPLE: 3 COIN TOSS AGAIN

- X – number of heads in three tosses

x (value)	Event $(X \leq x)$	# elements	$F_X(x)$
-1	$\emptyset$	0	0
0	{TTT}	1 (1 + 0)	$\frac{1}{8}$
1	{HTT, THT, TTH, TTT}	**4** (3+1)	$\frac{4}{8} = \frac{1}{2}$
2	{HHT, HTH, THH, **HTT, THT, TTH, TTT**}	7 (3 + **4**)	$\frac{7}{8}$
3			
4			

EXAMPLE: 3 COIN TOSS AGAIN

- X – number of heads in three tosses

x (value)	Event ($X \leq x$)	# elements	$F_X(x)$
-1	$\emptyset$	0	0
0	{TTT}	1 (1 + 0)	$\frac{1}{8}$
1	{HTT, THT, TTH, TTT}	4 (3+1)	$\frac{4}{8} = \frac{1}{2}$
2	{HHT, HTH, THH, HTT, THT, TTH, TTT}	**7** (3 + 4)	$\frac{7}{8}$
3	{HHH, **HHT, HTH, THH, HTT, THT, TTH, TTT**}	8 (1 + **7**)	1
4	S	8 (0 + 8)	1

EXAMPLE: 3 COIN TOSS AGAIN

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-1	$\emptyset$	0	0
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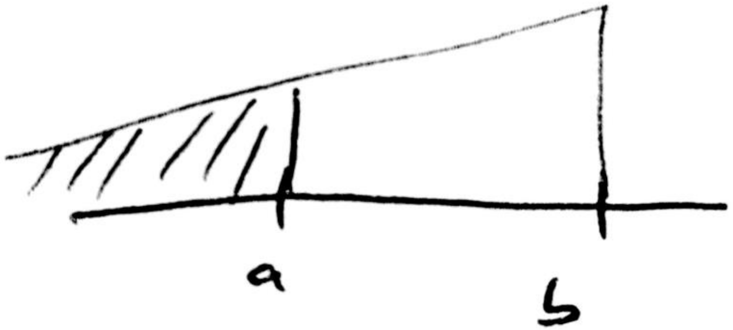
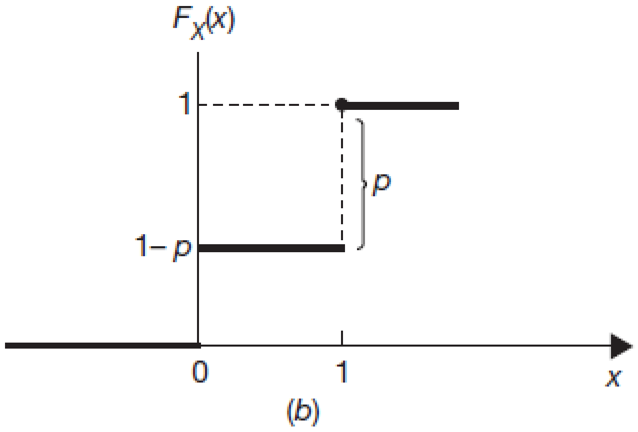
1	{HTT, THT, TTH, TTT}	4 (3+1)	$\frac{4}{8}=\frac{1}{2}$
2	{HHT, HTH, THH, HTT, THT, TTH, TTT}	7 (3 + 4)	$\frac{7}{8}$
3	{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}	8 (1 + 7)	1
4	S	8 (0 + 8)	1

PROBABILITIES FROM CDF

- Completely specify probabilities from a CDF
- 1) $P(a < X \leq b) = F_X(b) - F_X(a)$
  $= P(X \leq b) - P(X \leq a)$



- 2) $P(X > a) = 1 - F_X(a)$
- 3) $P(X < b) = F_X(b^-)$
  - $b^- = \lim_{0<\epsilon\to 0} b - \epsilon$
  - Approach from the left side

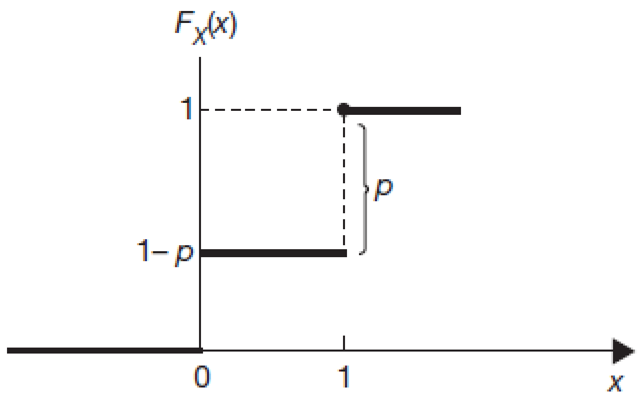


DISCRETE RVS AND PROBABILITY MASS FUNCTIONS

CHAPTER 2.4

DISCRETE RV

- X is RV with CDF $F_X(x)$ and $F_X(x)$ only changes in jumps (countably many) and is constant between jumps
  - Range of X contains a finite (countably infinite) number of points



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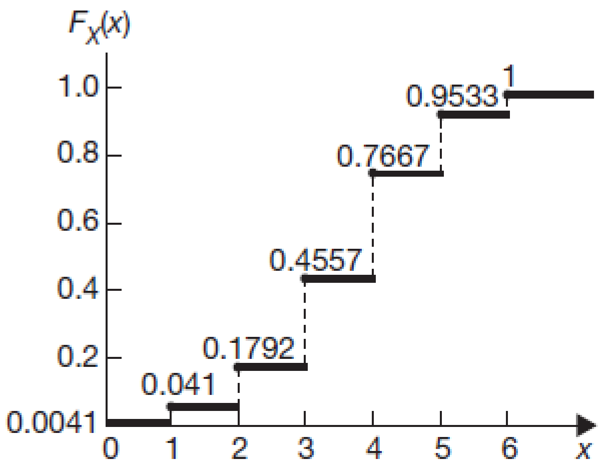

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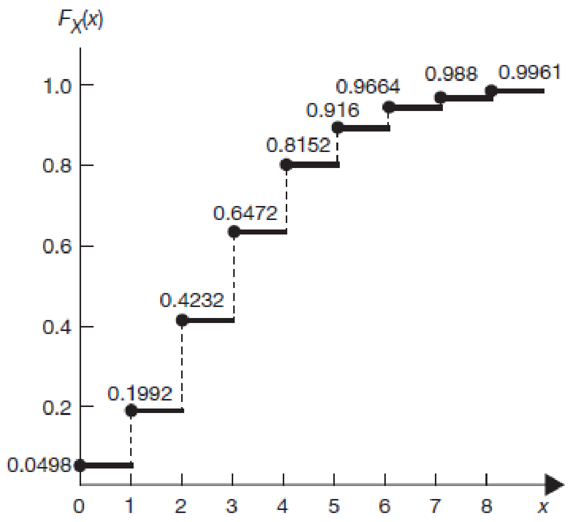

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PROBABILITY MASS FUNCTION (PMF)

- Given jumps in discrete RV @ points $x_1, x_2, \dots$ and $x_i < x_j$ for $i < j$
  - $p_X(x_i) = F_X(x_i) - F_X(x_{i-1})$
    $= P(X \leq x_i) - P(X \leq x_{i-1}) = P(X = x_i)$
- 3 Coin toss example

$\boldsymbol{x}$ (value)	# elements	$\boldsymbol{F_X(x)}$	$\boldsymbol{p_X(x)}$	Discussion
1	4 (3+1)	$\frac{4}{8} = \frac{1}{2}$	$p_X(1) = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$	<how much more needed from previous value>
2	7 (3 + 4)	$\frac{7}{8}$	$p_X(2) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$	3 extra outcomes
3	8 (1 + 7)	1	$p_X(3) = 1 - \frac{7}{8} = \frac{1}{8}$	1 extra outcome

PMF PROPERTIES

- 1) $0 \leq p_X(x_k) \leq 1 \quad\quad k = 1, 2, \ldots$ (finite set of values)
- 2) $p_X(x) = 0$ if $x \neq x_k$ (a value that cannot occur)
- 3) $\sum_k p_X(x_k) = 1$

- CDF from PMF
  - $F_X(x) = P(X \leq x) = \sum_{x_k \leq x} p_X(x_k)$
  - Accumulation of probability mass

CONTINUOUS RVS AND PROBABILITY DENSITY FUNCTIONS

CHAPTER 2.5

CONTINUOUS RV

- X is RV with CDF $F_X(x)$ continuous and derivative $\frac{dF_X(x)}{dx}$ exists
  - Range contains an interval of real numbers

- Note: $P(X = x) = 0$
  - There is zero probability for a particular continuous outcome → only over a range of values

PROBABILITY DENSITY FUNCTION (PDF)

- $f_X(x) = \frac{dF_X(x)}{dx}$ pdf of X
- Properties
- 1) $f_X(x) \geq 0$
- 2) $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- 3) $f_X(x)$ is piecewise continuous
- 4) $P(a < X \leq b) = \int_a^b f_X(x)dx$

$$= P(a \leq X \leq b)$$
$$= F_X(b) - F_X(a)$$

- CDF from PDF
  - $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(\xi)d\xi$

MEAN AND VARIANCE

CHAPTER 2.6

MEAN

- Expected value of RV X
- Discrete
  - $\mu_X = E[X] = \sum_k x_k p_X(x_k)$
- Continuous
  - $\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$

MOMENT

- n^{th} moment defined as:
- Discrete
  - $E[X^n] = \sum_k x_k^n \, p_X(x_k)$
- Continuous
  - $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$

VARIANCE

- $\sigma_X^2 = Var(X) = E[(X - E[X])^2]$
  - $E[.]$ – expected value operation
  - $E[X] = \mu_X$ - mean
- Discrete
  - $\sigma_X^2 = \sum_k (x - \mu_X)^2 p_X(x_k)$
- Continuous
  - $\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$

$$
\begin{aligned}
Var(X) &= E\left[(X - E[X])^2\right] \\
&= E[X^2 - 2X\mu_X + \mu_X^2] \\
&= E[X^2] - 2\mu_X E[X] + \mu_X^2 \\
&= E[X^2] - 2\mu_X^2 + \mu_X^2 \\
&= E[X^2] - \mu_X^2 \\
&= \underbrace{E[X^2]}_{\text{2nd moment}} - \underbrace{E^2[X]}_{\text{1st moment}}
\end{aligned}
$$

SOME SPECIAL DISTRIBUTIONS

CHAPTER 2.7

IMPORTANT DISTRIBUTIONS

- Model real-world phenomena
- Mathematically convenient specification for a probability distribution (usually pmf or pdf)

- Will examine similar discrete and continuous distributions
  - See book for technical details (e.g. mean/var)

BIG IDEA: PROBABILITY DISTRIBUTION

- Assign a probability to each of the possible outcomes of a random experiment
- Discrete
  - Probability mass function (pmf) – probability of each possible outcome
  - E.g. probability a roll of die will come up with a 3
- Continuous
  - Probability density function (pdf) – probability the outcome is within a range of values (interval)
  - E.g. probability that a 500 g package is between 490-510 g

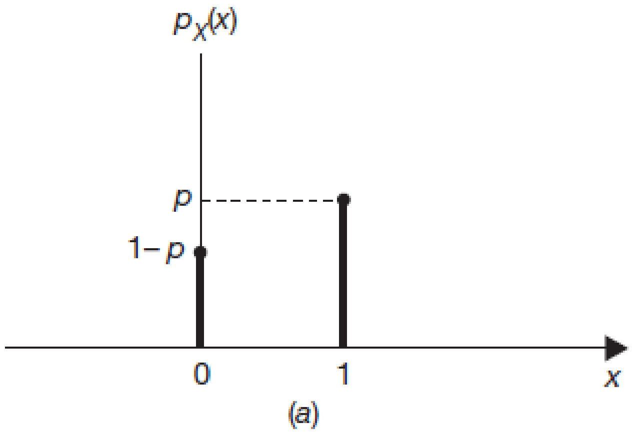
SPECIAL DISTRIBUTIONS

- Discrete
  - Bernoulli
  - Binomial
  - Geometric
  - Negative Binomial
  - Poisson
  - Uniform
- Continuous
  - Uniform
  - Exponential
  - Gamma
  - Normal

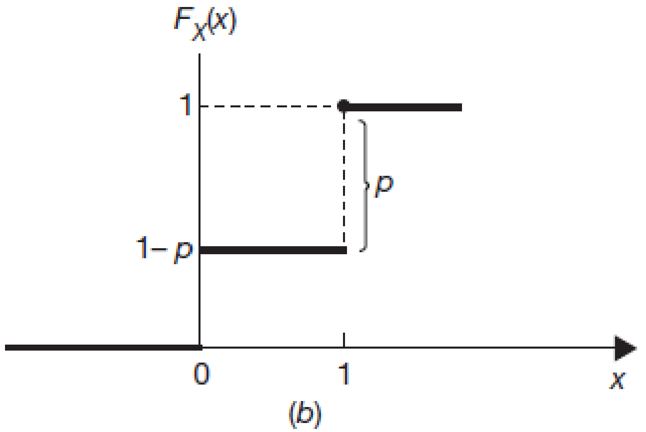
BERNOULLI DISTRIBUTION

- Binary RV with probability p of 1 (“success”) or $(1 - p)$ for failure
  - E.g. a coin flip with heads a “success” or “1” and tails a “failure” or “0”

- $p_X(k) = P(X = k) = p^k(1 - p)^{1-k}$
  - $0 < p < 1$ is probability of success
  - $(1 - p)$ is probability of failure
  - $k = 0,1$



pmf

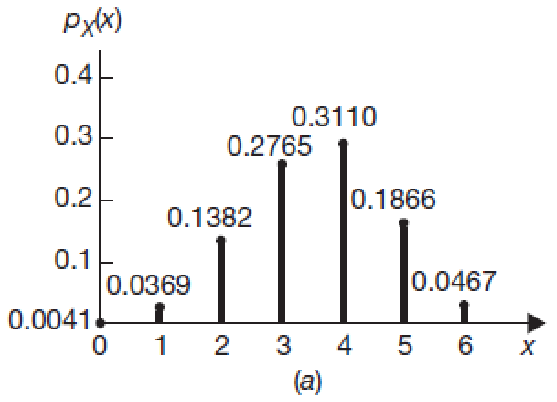


cdf

BINOMIAL DISTRIBUTION

- RV to count the number of successes with n independent Bernoulli trials

- $p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
  - $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ - n choose k
  - Number of ways to get k successes (heads) in n trials (coin tosses)



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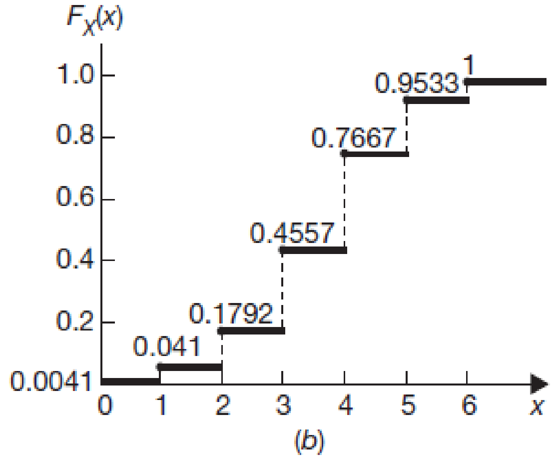

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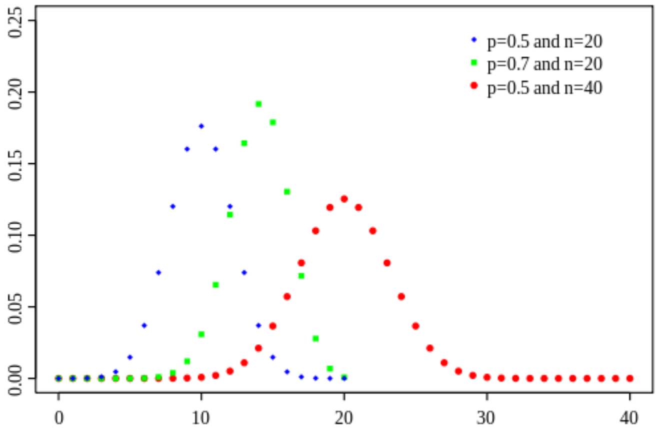
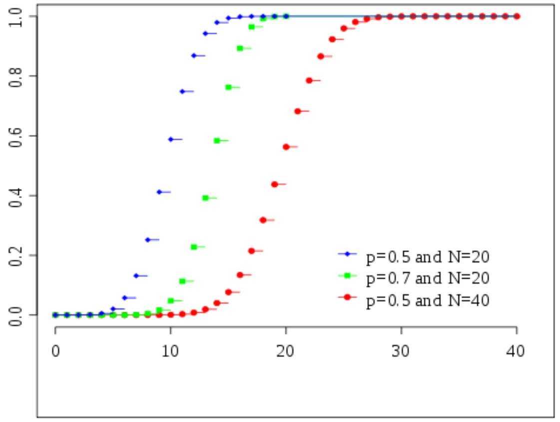
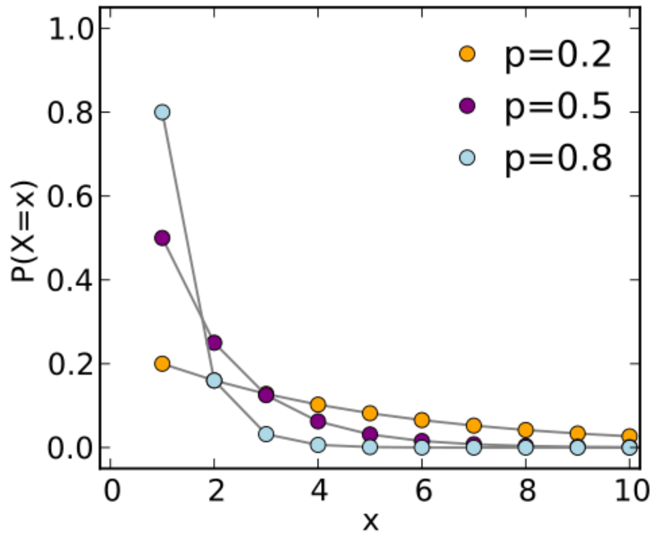




GEOMETRIC DISTRIBUTION

- Sequence of Bernoulli trials observed until first success
- $p_X(x) = P(X = x) = (1 - p)^{x-1}p$



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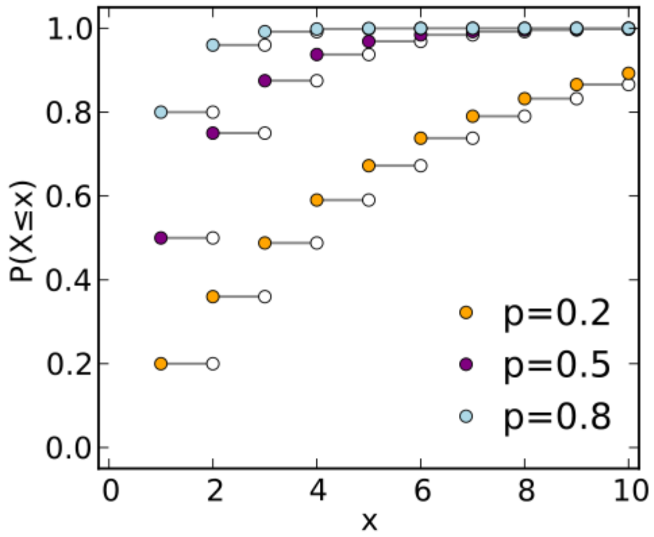

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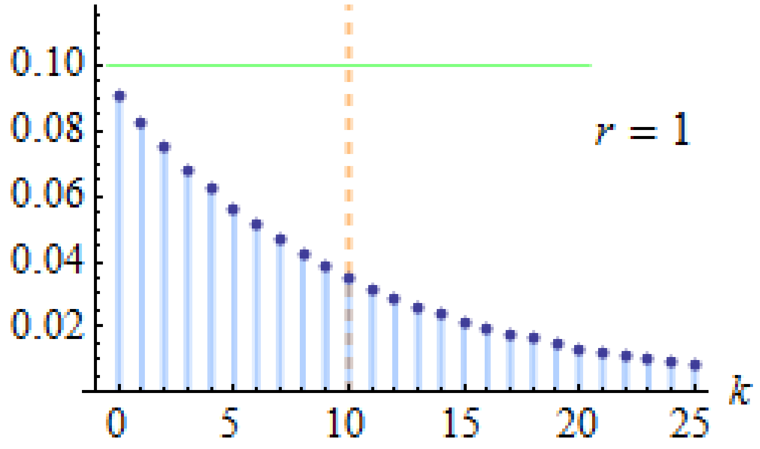
NEGATIVE BINOMIAL DISTRIBUTION

- Number of trials until kth success in sequence of Bernoulli trials

- $p_X(x) = P(X = x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$

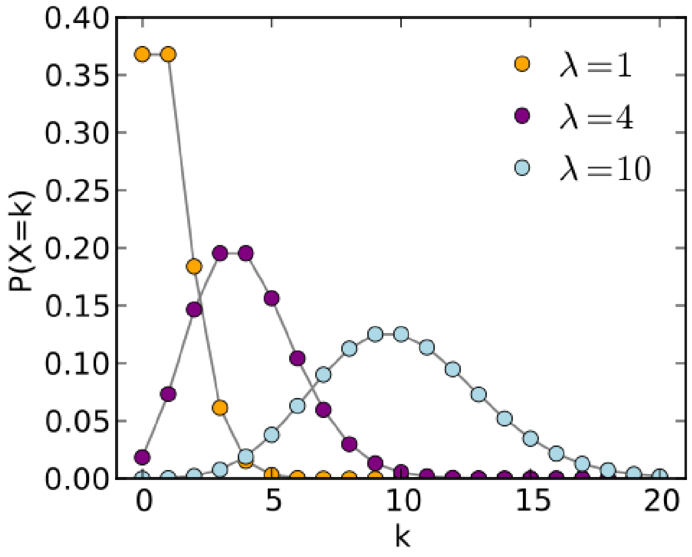


Note: parameter $r = x - k$

pmf

POISSON DISTRIBUTION

- The number of events occurring in a fixed interval (time or space) given a known event average rate λ
- $p_X(k) = P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$



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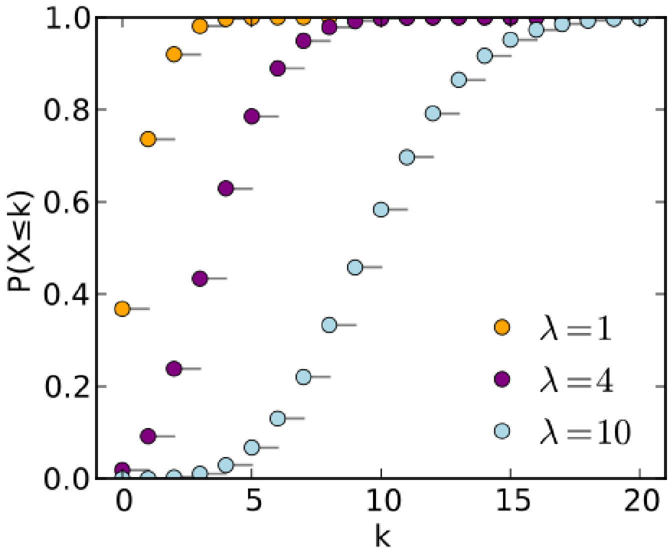

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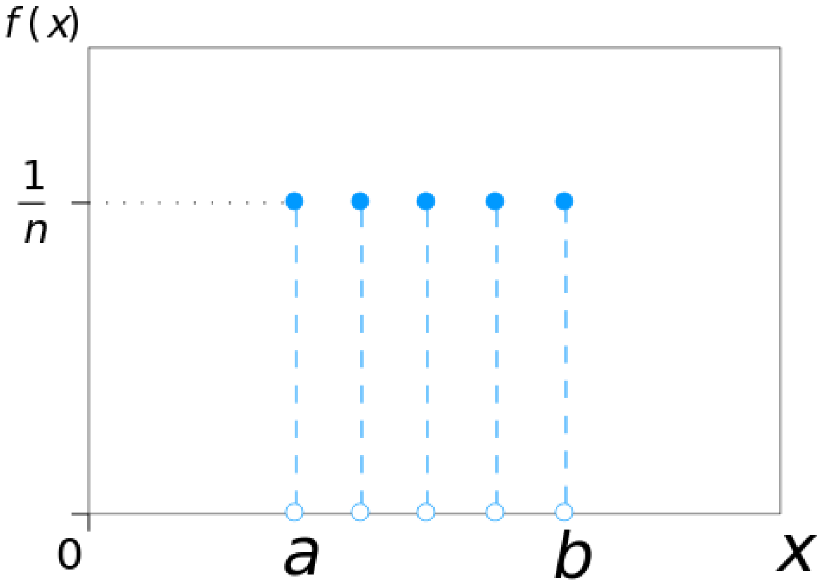
DISCRETE UNIFORM DISTRIBUTION

- Equally likely outcomes
- $p_X(x) = P(X = x) = \frac{1}{n}$



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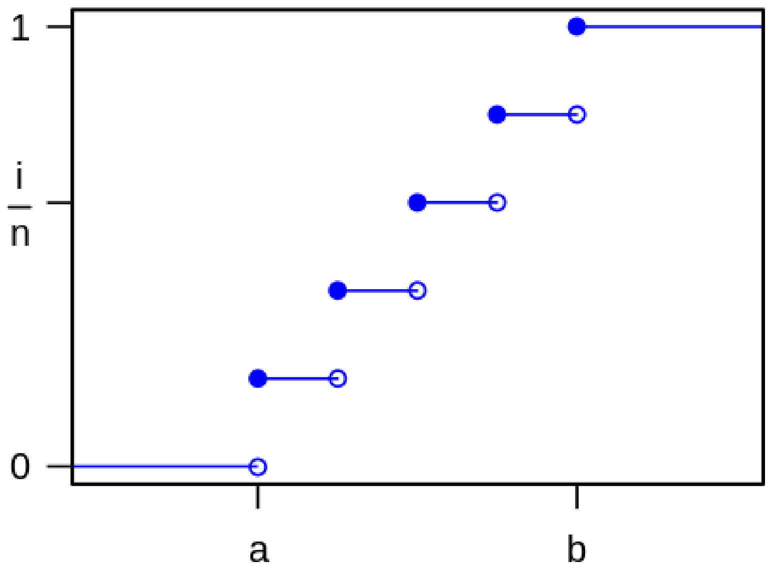

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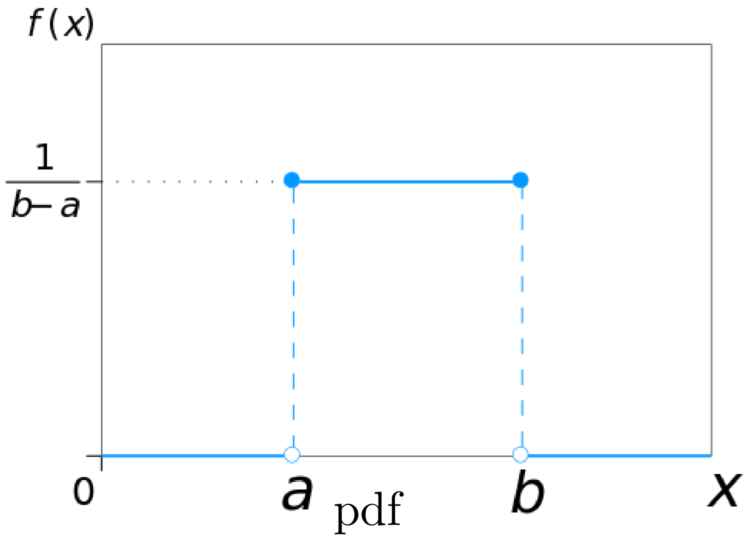
CONTINUOUS UNIFORM DISTRIBUTION

- Often used when no prior knowledge and equally likely values in a range

- $f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & else \end{cases}$



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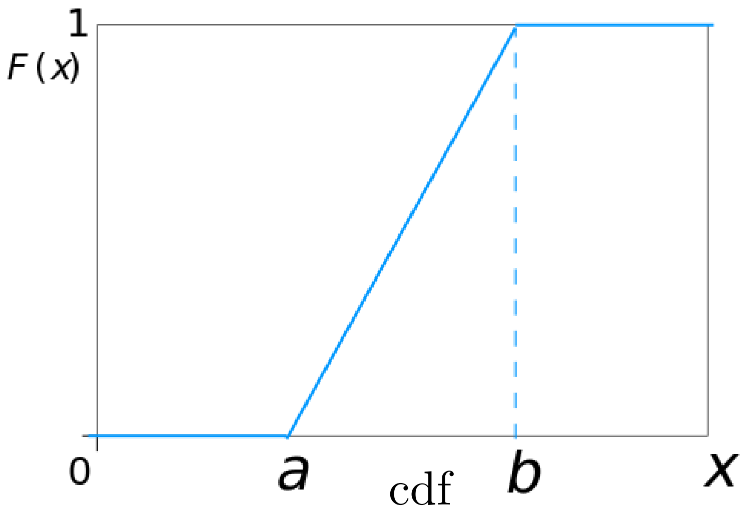

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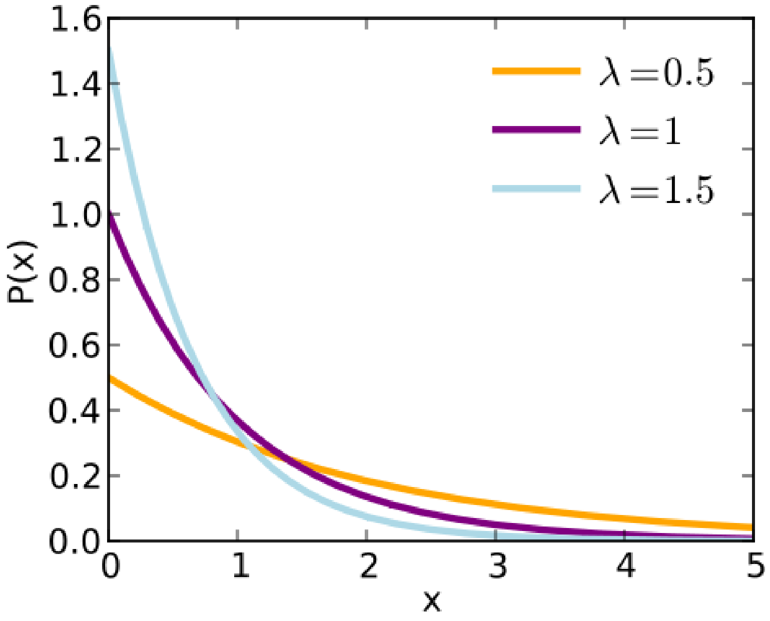
EXPONENTIAL DISTRIBUTION

- Time decay – memoryless property
- $f_X(x) = \lambda e^{-\lambda x} \quad x > 0$



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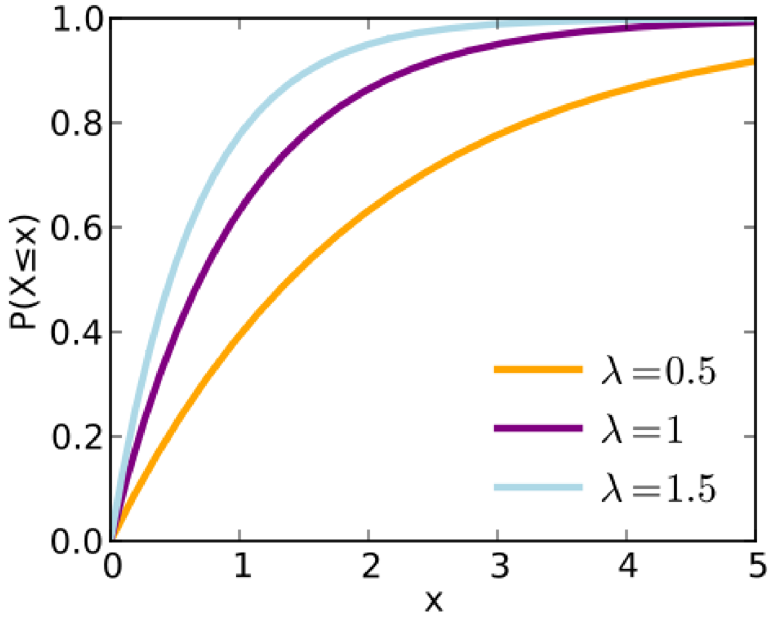

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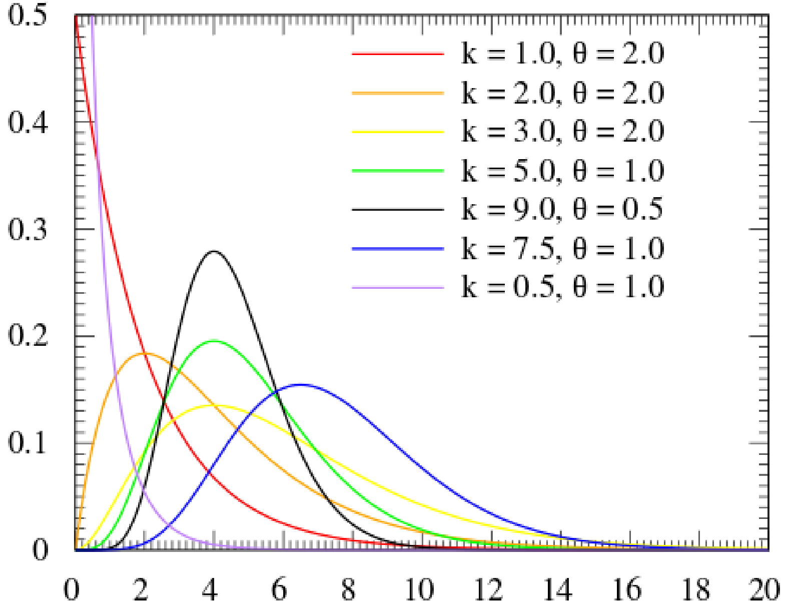
GAMMA DISTRIBUTION

- $f_X(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \quad x > 0$



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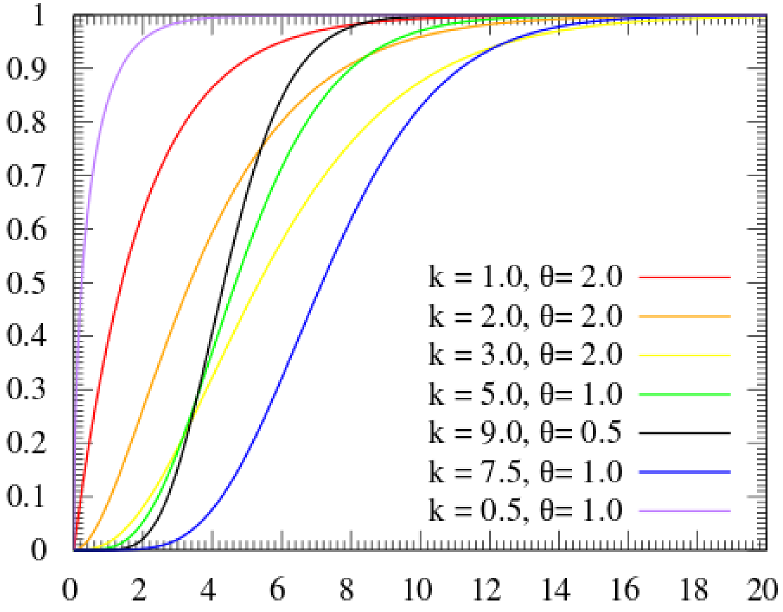

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α – shape parameter

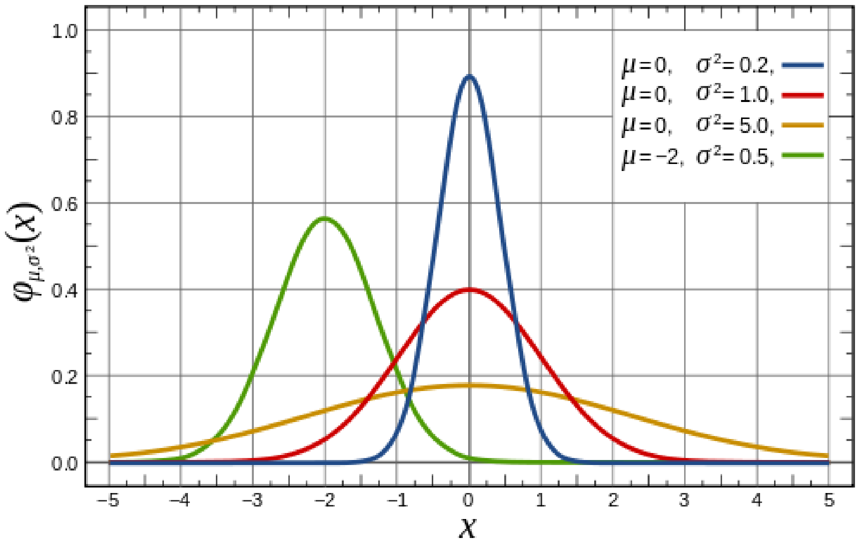
$\theta = \frac{1}{\lambda}$ rate parameter (inverse scale)

NORMAL (GAUSSIAN) DISTRIBUTION

- Models many natural phenomena

- $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$

Shorthand notation
$X \sim N(\mu, \sigma^2)$



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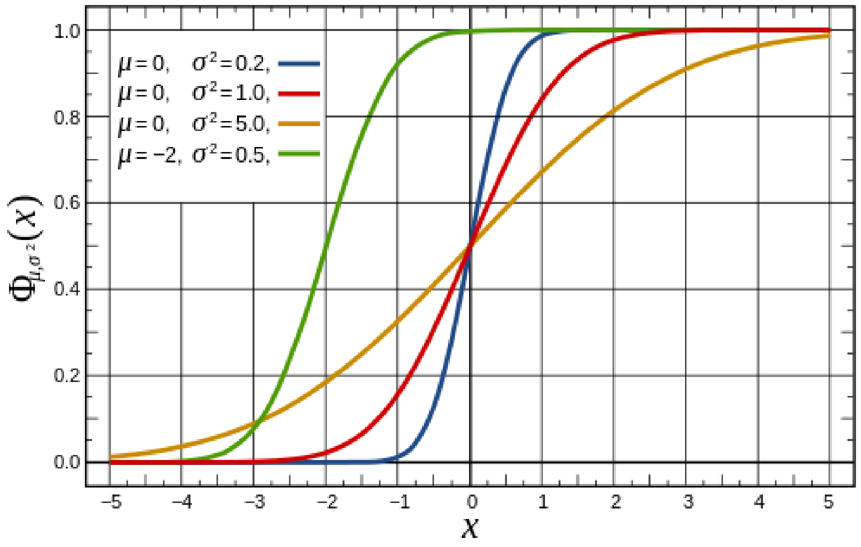

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CONDITIONAL DISTRIBUTIONS

CHAPTER 2.8

CONDITIONAL DISTRIBUTIONS

- Remember $P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$
- Conditional CDF
  - $F_X(x|B) = P(X \leq x|B) = \frac{P\{(X \leq x) \cap B\}}{P(B)}$
- Conditional PMF
  - $p_X(x_k|B) = P(X = x_k|B) = \frac{P\{(X = x_k) \cap B\}}{P(B)}$
- Conditional PDF
  - $f_X(x|B) = \frac{d}{dx} F_X(x|B)$