EE361: SIGNALS AND SYSTEMS II

CH5: DISCRETE TIME FOURIER TRANSFORM

FOURIER TRANSFORM DERIVATION

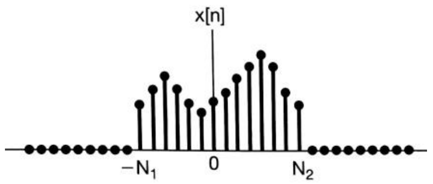
CHAPTER 5.1-5.2

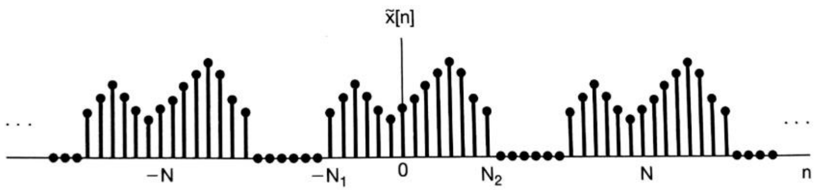
FOURIER SERIES REMINDER

- Previously, FS allowed representation of a periodic signal as a linear combination of harmonically related exponentials
  - $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$ $\qquad a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}\, dt$
    - $\omega_0 = \frac{2\pi}{N}$
- Would like to extend this (Transform Analysis) idea to aperiodic (non-periodic) signals

DT FOURIER TRANSFORM DERIVATION

- Intuition (same idea as CTFT):
- Consider a finite signal $x[n]$
- Periodic pad to get periodic signal $\tilde{x}[n]$
- Find FS representation of $\tilde{x}[n]$
- Analyze FS as $N \to \infty$ ($\omega_0 \to 0$) to get DTFT
  - Note DTFT is discrete in time domain – continuous in frequency domain
- Envelope $X(e^{j\omega})$ of normalized FS coefficients $\{a_k N\}$ defines the DTFT (spectrum of $x[n]$)





DT FOURIER TRANSFORM PAIR

- $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ synthesis eq (inverse FT)
- $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ analysis eq (FT)
  - DTFT is discrete in time – continuous in frequency

- Notice the DTFT $X(e^{j\omega})$ is period with period 2π

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\underbrace{e^{-j2\pi n}}_{=1} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega})$$

DTFT CONVERGENCE

- The FT converges if
  - $\sum_n |x[n]| < \infty$ absolutely summable
  - $\sum_n |x[n]|^2 < \infty$ finite energy

- iFT has not convergence issues because $X\left(e^{j\omega}\right)$ is periodic
  - Integral is over a finite 2π period (similar to FS)

FT OF PERIODIC SIGNALS

- Important property
  - $x[n] = e^{jk\omega_0 n} \leftrightarrow X(j\omega) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0 - 2\pi l)$
    - Impulse at frequency $k\omega_0$ and 2π shifts
- Transform pair
- $\sum_{k=<N>} a_k e^{jk\omega_0 n} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$
  - Each a_k coefficient gets turned into a delta at the harmonic frequency

DTFT PROPERTIES AND PAIRS

CHAPTER 5.3-5.6

PROPERTIES/PAIRS TABLES

- Most often will use Tables to solve problems
- Table 5.1 pg 391 – DTFT Properties
- Table 5.2 pg 392 – DTFT Transform Pairs

NOTEWORTHY PROPERTIES

- Periodicity – $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$
- Time shift – $x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$
- Frequency/phase shift – $e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
- Convolution – $x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$
- Multiplication – $x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
  - Notice this is an integral over a single period → periodic convolution $\frac{1}{2\pi}X(e^{j\omega}) * Y(e^{j\omega})$

NOTEWORTHY PAIRS I

- Decaying exponential
  - $h[n] = a^n u[n] \qquad |a| < 1$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

- Magnitude response

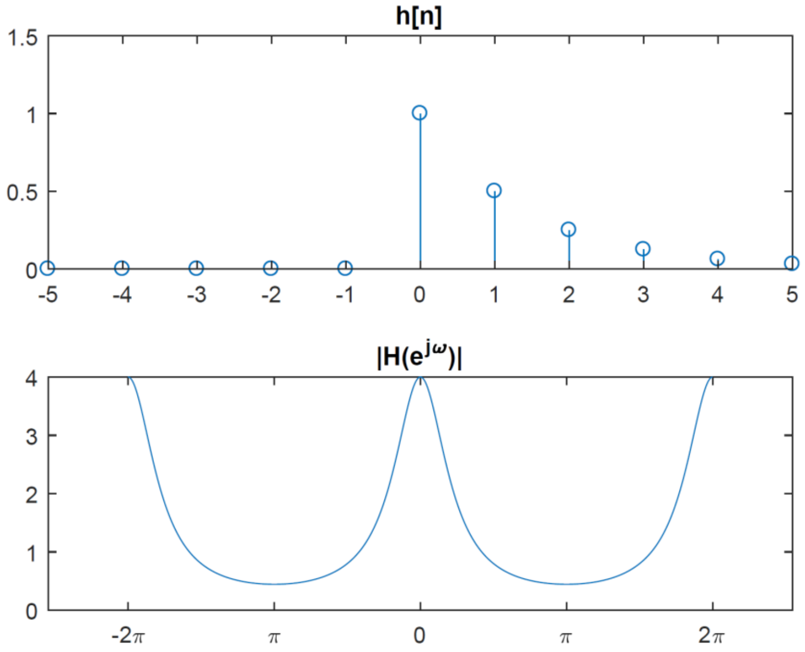
$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \left(\frac{1}{1 - ae^{-j\omega}}\right)\left(\frac{1}{1 - ae^{+j\omega}}\right)$$

$$= \frac{1}{1 + a^2 - ae^{j\omega} - ae^{-j\omega}}$$

$$= \frac{1}{1 + a^2 - 2a\cos(\omega)}$$

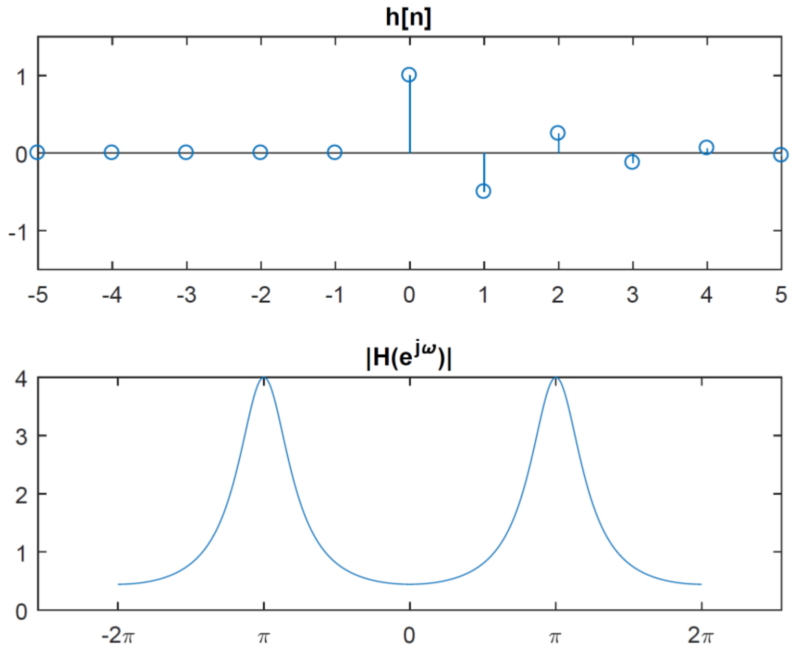
DECAYING EXPONENTIAL

- $0 < a < 1$



- Lowpass filter

- $-1 < a < 0$



- Highpass filter

NOTEWORTHY PAIRS II

- Impulse
  - $x[n] = \delta[n] \leftrightarrow X(e^{j\omega}) = \sum_n \delta[n] e^{-j\omega n} = \sum_n \delta[n] e^{-j\omega(0)} = \sum_n \delta[n] = 1$
  - $x[n] = \delta[n - n_0] \leftrightarrow X(e^{j\omega}) = \sum_n \delta[n - n_0] e^{-j\omega n} = \sum_n \delta[n - n_0] e^{-j\omega n_0} = e^{-j\omega n_0}$
- Rectangle pulse
  - $x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases} \leftrightarrow X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \frac{\sin\left(\omega\left(\frac{2N_1+1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$
- Periodic signal
  - $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \leftrightarrow X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$
    - One period of a_k copied

DTFT AND LTI SYSTEMS

CHAPTER 5.8

GENERAL DIFFERENCE EQUATION SYSTEM

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Take FT of both sides

$$\sum_{k=0}^{N} a_k \mathcal{F}\{y[n-k]\} = \sum_{k=0}^{M} b_k \mathcal{F}\{x[n-k]\}$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$Y(e^{j\omega}) \left[\sum_{k=0}^{N} a_k e^{jk\omega}\right] = X(^{j\omega}) \left[\sum_{k=0}^{M} b_k e^{-jk\omega}\right]$$

- Solve for frequency response

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

- Rational form – ratio of polynomials in $e^{-j\omega}$
- Best solved using partial fraction expansion (Appendix A)
  - Note special heavy-side cover-up approach for repeated root

LTI SYSTEM APPROACH

- Same techniques as in continuous case

- $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
- Partial fraction expansion
- Inverse FT with tables