EE361: SIGNALS AND SYSTEMS II

CH4: CONTINUOUS TIME FOURIER TRANSFORM

FOURIER TRANSFORM DERIVATION

CHAPTER 4.1

FOURIER SERIES REMINDER

- Previously, FS allowed representation of a periodic signal as a linear combination of harmonically related exponentials
  - $x(t) = \sum_k a_k e^{jk\omega_0 t}$ $\qquad a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$
- Would like to extend this (Transform Analysis) idea to aperiodic (non-periodic) signals

CT FOURIER TRANSFORM DERIVATION I

- Intuition:
- Consider a periodic signal with period T
- Let $T \to \infty$
  - Infinite period → no longer periodic signal
- Results in $\omega_0 = \frac{2\pi}{T} \to 0$
  - Zero frequency space between "harmonics" → differential $d\omega$
- Envelope (like we saw with rectangle wave/sinc) defines the CTFT

CT FOURIER TRANSFORM DERIVATION II

- Will skip derivation for now
- Please see details in the book

CT FOURIER TRANSFORM PAIR

- $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ synthesis eq (inverse FT)
- $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ analysis eq (FT)

- Denote
  - $x(t) \leftrightarrow X(j\omega)$
  - $X(j\omega) = \mathcal{F}\{x(t)\}$ $x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$

CTFT CONVERGENCE

- There are conditions on signal $x(t)$ for FT to exist
- Finite energy (square integrable)
  - $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$
- Dirichlet Conditions
  - We will not cover; see pg 290 for more discussion

CTFT FOR PERIODIC SIGNALS

CHAPTER 4.2

FT OF PERIODIC SIGNALS

- Derived FT by assuming a periodic padding of aperiodic signal $x(t)$
- What happens for FT of a periodic signal?
  - Note: periodic signal will not have finite energy
  - Cannot evaluate FT integral directly

PERIODIC FT DERIVATION I

- From derivation of FT, $X(j\omega)$ is the envelope of Ta_k
  - FS coefficients are scaled samples of $X(j\omega)$
- Assume $x(t)$ is periodic $[x(t) = x(t+T)]$
- Then, $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, with $\omega_0 = \frac{2\pi}{T}$
- Plug into FT integral and solve
- Will not solve for now on slides → see book

PERIODIC FT DERIVATION II

- Important property
  - $x(t) = e^{jk\omega_0 t} \leftrightarrow X(j\omega) = 2\pi\delta(\omega - k\omega_0)$

- Transform pair
- $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$
  - Each a_k coefficient gets turned into a delta at the harmonic frequency

FT OF SINUSOIDAL SIGNALS

- FT of periodic signals is important because of sinusoidal signals (cannot solve using FT integral)
  - Can use insight of complex exponential ↔ shifted delta from periodic FT derivation
- Important examples

$$x(t) = \sin \omega_0 t \xrightarrow{\text{FS}} a_1 = \frac{1}{2j} \qquad \Rightarrow \quad X(j\omega) = \frac{2\pi}{-2j}\delta(\omega + \omega_0) + \frac{2\pi}{2j}\delta(\omega - \omega_0)$$

$$a_{-1} = -\frac{1}{2j} \qquad = -\frac{\pi}{j}\delta(\omega + \omega_0) + \frac{\pi}{j}\delta(\omega - \omega_0)$$

$$x(t) = \cos \omega_0 t \xrightarrow{\text{FS}} a_1 = \frac{1}{2} \qquad \Rightarrow \quad X(j\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$

$$a_{-1} = \frac{1}{2}$$

CTFT PROPERTIES AND PAIRS

CHAPTER 4.3-4.6

PROPERTIES TABLE 4.1 (PG 328)

- Linearity
  - $x(t) \leftrightarrow X(j\omega)$
  - $y(t) \leftrightarrow Y(j\omega)$
  - $ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$
- Time shifting
  - $x(t - t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$
  - Note, this is a phase shift of $X(j\omega)$
- Conjugation
  - $x^*(t) \leftrightarrow X^*(-j\omega)$
    - Remember: conjugation switches sign of imaginary part
- Time scaling
  - $x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$
- Differentiation in time
  - $\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)$
- Integration in time
  - $\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$

CONVOLUTION/MULTIPLICATION PROPERTIES

- Convolution
  - $y(t) = h(t) * x(t) \leftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$
- Multiplication
  - $r(t) = s(t)p(t) \leftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P\big(j(\omega - \theta)\big)d\theta$
  - $R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega)$
- Dual properties – convolution $\leftrightarrow$ multiplication

FT PAIRS TABLE 4.2 (PG 329)

- Be sure to bookmark this table (right next to Table 4.1 Properties)
- Note in particular some very useful pairs that aren't typical
  - $te^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^2}$ repeated root
  - $u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$

CTFT AND LTI SYSTEMS

CHAPTER 4.7

FIRST-ORDER EXAMPLE

- Find impulse response $h(t)$

$$\frac{d}{dt}y(t) + ay(t) = x(t)$$

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$Y(j\omega)\left[j\omega + a\right] = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a + j\omega}$$

$$h(t) = \mathcal{F}^{-1}\left\{\frac{1}{a + j\omega}\right\} = e^{-at}u(t)$$

LTI SYSTEM ANALYSIS

- Note for $H(j\omega)$ to exist, the LTI system must have impulse response $h(t)$ that satisfies stability conditions

- FT is only for the analysis of stable LTI systems
  - For not stable systems, use Laplace Transform in Ch9

GENERAL DIFFERENTIAL EQUATION SYSTEM

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- Take FT of both sides

$$\sum_{k=0}^{N} a_k \mathcal{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{F}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)\right]$$

- Solve for frequency response

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

- Rational form – ratio of polynomials in $j\omega$
- Best solved using partial fraction expansion (Appendix A)