EE361: SIGNALS AND SYSTEMS II

CH6: ANALYSIS AND PROCESSING OF RANDOM PROCESSES

NOTE ON CONTENT

- We will only cover
  - 6.3 Power Spectral Densities
  - 6.4 White Noise
  - 6.5 Response of Linear Systems to Random Inputs

INTRODUCTION

- This chapter finally connects the second half of ee361 (probability, random variables, random processes) to ee360 and first half ee361 (LTI systems)
  - Explain how to analyze an LTI system with stochastic input

STOCHASTIC CORRELATION REMINDER

- Autocorrelation
  - $R_X(\tau) = R_{XX}(\tau) = E[X(t)X(t+\tau)]$
- Properties
  - 1) $R_X(-\tau) = R_X(\tau)$
  - 2) $|R_X(\tau)| \leq R_X(0)$
  - 3) $R_X(0) = E[X^2(t)] \geq 0$

- Cross correlation
  - $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$
- Properties
  - 1) $R_{XY}(-\tau) = R_{YX}(\tau)$
  - 2) $|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$
  - 3) $|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$

- If $R_{XY}(\tau) = 0 \ \forall\tau \Rightarrow$ mutually orthogonal

POWER SPECTRAL DENSITY (POWER SPECTRUM)

- Describes how variance of data is distributed over frequency
  - Indicates which frequency variations are strong (more energy)
- Defines as Fourier Transform of autocorrelation
  - $S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$ ← FT
  - $R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) e^{j\omega\tau} d\omega$ ← iFT

PROPERTIES OF PSD

- $S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$
- $R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) e^{j\omega\tau} d\omega$

- 1) $S_X(\omega) \geq 0$ and real
- 2) $S_X(-\omega) = S_X(\omega)$
- 3) $E[X^2(t)] = R_X(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$
- For discrete
- 4) $S_X(\Omega + 2\pi) = S_X(\Omega)$ 2π periodic

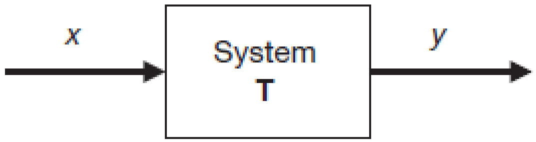
CROSS PSD (CROSS POWER SPECTRUM)

- $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$ $\leftarrow \mathfrak{I}\{R_{XY}(\tau)\}$
- $R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$ $\leftarrow \mathfrak{I}^{-1}\{S_{XY}(\omega)\}$
- Properties
  - 1) $S_{XY}(\omega) = S_{YX}(-\omega)$
  - 2) $S_{XY}(-\omega) = S_{XY}^{*}(\omega)$
  - Note: $S_{XY}(\omega)$ is complex valued, discrete version is also 2π periodic

WHITE NOISE

- The white noise RP $w(t)$, is a wide-sense stationary (WSS) zero-mean CT random process with:
  - $R_W(\tau) = \sigma^2\delta(\tau)$
  - $S_W(\omega) = \int_{-\infty}^{\infty} \sigma^2\delta(\tau)\, e^{-j\omega\tau} d\tau = \sigma^2$
- White noise process has a constant power spectrum
  - All frequencies are equally represented → "white" signal
- Note: the avg. power of $w(t)$ is not finite
  - DT has finite power for $w(n)$ because $S_W(\Omega) = \sigma^2$ is over a 2π interval (period)

LTI RESPONSE TO RANDOM INPUTS

- Remember
  - 

    $y = T\{x\} = h * x$ for LTI system
- CT case
  - $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$
- DT case
  - $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
- Note: we are using here notation from Oppenheim and Schafer

CT LTI SYSTEM WITH RANDOM INPUTS I

- When input is RP $\{X(t), t \in T_X\}$, then output is also a RP $\{Y(t), t \in T_Y\}$
  - $\{Y(t), t \in T_Y\} = T\{X(t), t \in T_X\}$
    - Poor notation: system T operating on RP $X(t)$
- For a realization (specific input signal)
  - $y_i(t) = T\{x_i(t)\}$
    - Our traditional study from ee360 and first half of ee361

CT LTI SYSTEM WITH RANDOM INPUTS II

- Given LTI system with random input (can't specifically give output $Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$)
- Specify stochastically
  - $E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau$
  - $R_Y(t,s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(t-\alpha, s-\beta)d\alpha d\beta$

CT LTI SYSTEM WITH RANDOM INPUTS III

- For WSS $X(t)$
  - $E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau$

    $$= \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau = \mu_X H(j\omega)|_{\omega=0}$$

    - Since $E[X(t-\tau)]$ is a constant and $H(j\omega) = \Im\{h(t)\}$
  - $R_Y(t, t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(\tau+\alpha-\beta)d\alpha d\beta = R_Y(\tau)$
- $Y(t)$ is WSS since:
  - 1) $E[Y(t)] = \mu_X H(0)$ ← constant
  - 2) $R_Y(t, t+\tau) = R_Y(t, s) = R_Y(\tau)$

WSS LTI PSD

- $S_Y(\omega) = \int_{-\infty}^{-\infty} R_Y(\tau)e^{-j\omega\tau}d\tau = |H(\omega)|^2 S_X(\omega)$
  - $S_Y(\omega)$ - output psd
  - $|H(\omega)|^2$ - magnitude squared of frequency response
  - $S_X(\omega)$ - input psd
- $R_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 S_X(\omega)e^{j\omega\tau}d\tau$
- Average power
  - $E[Y^2(t)] = R_Y(0) = \frac{1}{2\pi}\int_{\infty}^{\infty}|H(\omega)|^2 S_X(\omega)d\omega$

DT LTI RESPONSE

- Very similar results to CT case
  - $Y(n) = \sum_{i=-\infty}^{\infty} h(i) X(n-i)$
  - $R_Y(n,m) = \sum_i \sum_l h(i) h(l) R_X(n-i, m-l)$
- For WSS $X(n)$
  - $E[Y(n)] = \mu_X H(0)$
  - $R_Y(n,m) = R_Y(n, n+k) = R_Y(k)$
  - $S_Y(\Omega) = |H(\Omega)|^2 S_X(\Omega)$

RECAP

- Since random input gives random output, analyze LTI system stochastically
  - Power spectrum is FT of (auto)correlation
- Will focus on WSS processes
  - 1^{st}-order specification
    - Expected output: $E[Y(t)] = \mu_X H(j\omega)|_{\omega=0}$
  - 2^{nd}-order specification
    - Autocorrelation: $R_Y(\tau) = \mathfrak{F}^{-1}\{S_Y(\omega)\} = \mathfrak{F}^{-1}\{|H(j\omega)|^2 S_X(\omega)\}$

EXAMPLE 6.26

- $X(t)$ WSS and $R_X(\tau) = e^{-a|\tau|}$ for input
- $h(t) = e^{-bt}u(t)$
- Find autocorrelation of output $Y(t)$

- Will do in discussion