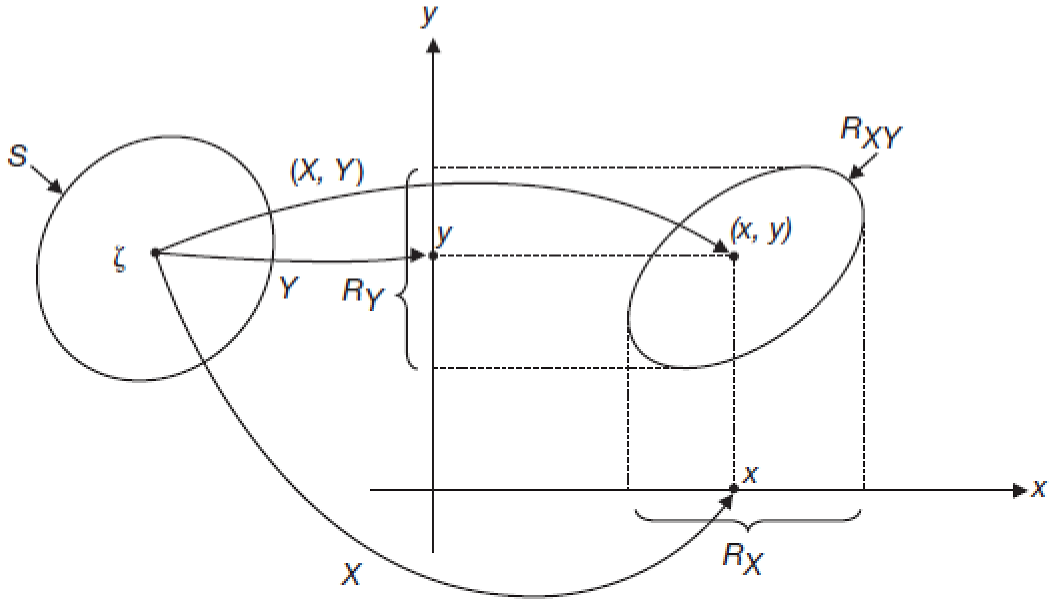
EE361: SIGNALS AND SYSTEMS II

CH3: MULTIPLE RANDOM VARIABLES

BIVARIATE RANDOM VARIABLES

- A pair of RV (X, Y) that associates two real numbers with every element in S
  - Two-dimensional random vector
- Function that maps outcome ξ to a point in the (x,y)-plane
- Range of (X, Y)
- $R_{XY} = \{(x, y); \xi \in S \text{ and } X(\xi) = x, Y(\xi) = y\}$



BIVARIATE RV TYPES

- Bivariate discrete RV – both X, Y discrete
- Bivariate continuous RV – both X, Y continuous
- Bivariate mixed RV – one discrete other continuous

- In this class will primarily focus on either bivariate discrete or continuous, not mixed

JOINT DISTRIBUTION FUNCTIONS (CDF)

- $F_{XY}(x, y) = P(X \leq x, Y \leq y)$
  $= P(A \cap B)$
  - Event A: $(X \leq x)$; Event B: $(Y \leq y)$
- Formally, event $(X \leq x, Y \leq y)$ = event $(A \cap B)$
  - $A = \{\xi \in S; X(\xi) \leq x\}$
    - $P(A) = F_X(x)$
  - $B = \{\xi \in S; Y(\xi) \leq y\}$
    - $P(B) = F_Y(y)$
- Independent RV
  - $F_{XY}(x, y) = F_X(x)F_Y(y)$
    $= P(A)P(B)$
- Properties – same general idea as for single RV

MARGINAL DISTRIBUTION

- Given joint CDF,
  - $F_X(x) = F_{XY}(x, \infty)$
  - $F_Y(y) = F_{XY}(\infty, y)$
- These are the distribution taking into account all values of the other RV
  - E.g. marginalizing/removing the effects/dependence on one variable
- Result comes from observation
  - $\lim_{y\to\infty}(X \le x, Y \le y) = (X \le x, Y \le \infty) = (X \le x)$
  - The condition $(Y \le \infty)$ is always satisfied

JOINT PMF

- Let (X, Y) be discrete RV with values (x_i, y_j) for an allowable set of integers i, j
  - $p_{XY}(x_i, y_j) = P(X = x_i, Y = y_j)$
- Properties
  - 1) $0 \le p_{XY}(x_i, y_j) \le 1$
  - 2) $\sum_{x_i} \sum_{y_j} p_{XY}(x_i, y_j) = 1$
  - 3) $P[(X, Y) \in A] = \sum \sum_{(x_i, y_j) \in R_A} p_{XY}(x_i, y_j)$
    - Points $(x_i, y_j) \in R_A$ are in range space corresponding to event A
- CDF from PMF
  - $F_{XY}(x, y) = \sum_{x_i \le x} \sum_{y_j \le y} p_{XY}(x_i, y_j)$

MARGINAL PMF

- $P(X = x_i) = P_X(x_i) = \sum_{y_j} p_{XY}(x_i, y_j)$
  - Summation is over all possible $Y = y_j$ values
  - Marginalize by removing influence of RV Y
- $P(Y = y_j) = P_Y(y_j) = \sum_{x_i} p_{XY}(x_i, y_j)$
- Independence:
  - $P_{XY}(x_i, y_j) = p_X(x_i) p_Y(y_j)$

JOINT PDF

- (X, Y) is a continuous bivariate RV with CDF $F_{XY}(x, y)$
  - $f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$
  - $F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\xi, \eta) d\eta d\xi$
- Properties:
  - 1) $f_{XY}(x,y) \geq 0$
  - 2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) dxdy = 1$
  - 4) $P[(X,Y) \in A] = \int \int_{R_A} f_{XY}(x,y) dxdy$
  - 5) $P(a < X \leq b, c < Y \leq d) = \int_{c}^{d} \int_{a}^{b} f_{XY}(x,y) dxdy$

MARGINAL PDF

- $F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(\xi, \eta) d\eta d\xi$
  - Integrate/marginalize over full range/all values of y
- $f_X(x) = \frac{dF_X(x)}{dx} = \int_{-\infty}^{\infty} f_{XY}(x, \eta) d\eta = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$
- $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$
- Independence:
  - $F_{XY}(x, y) = F_X(x) F_Y(y)$
  - $f_{XY}(x, y) = f_X(x) f_Y(y)$

CONDITIONAL PMF

- (X,Y) discrete bivariate RV with joint PMF $p_{XY}(x_i, y_j)$
  - $p_{Y|X}(y_j|x_i) = \frac{p_{XY}(x_i,y_j)}{p_X(x_i)}$, $p_X(x_i) > 0$
  - Conditional PMF of Y given X $(= x_i)$ → probability of $Y = y_j$ knowing that $X = x_i$
- Properties
  - 1) $0 \leq p_{Y|X}(y_j|x_i) \leq 1$
  - 2) $\sum_{y_j} p_{Y|X}(y_j|x_i) = 1$
- Independence
  - $p_{Y|X}(y_j|x_i) = p_Y(y_j)$ and $p_{X|Y}(x_i|y_j) = p_X(x_i)$

CONDITIONAL PDF

- (X, Y) continuous bivariate RV with joint PMF $f_{XY}(x, y)$
  - $f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}, \qquad f_X(x) > 0$
  - Conditional PDF of Y given X $(= x)$
- Properties
  - 1) $f_{Y|X}(y|x) \geq 0$
  - 2) $\int_{-\infty}^{\infty} f_{Y|X}(y|x)dy = 1$
- Independence
  - $f_{Y|X}(y|x) = f_Y(y) \quad$ and $\quad f_{X|Y}(x, y) = f_X(x)$

(k,n)th MOMENT

- $m_{kn} = E[X^k Y^n]$
  - Discrete: $m_{kn} = \sum_{y_j} \sum_{x_i} x_i^k y_j^n p_{XY}(x_i, y_j)$
  - Continuous: $m_{kn} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^n f_{XY}(x, y) dx dy$
- Note: $m_{10} = E[X] = \mu_X$ and $m_{01} = E[Y] = \mu_Y$

$$\mu_X = \sum_{y_j} \sum_{x_i} x_i y_j^0 p_{XY}(x_i, y_j)$$
$$= \sum_{x_i} x_i \underbrace{\sum_{y_j} p_{XY}(x_i, y_j)}_{\text{marginalize}} = \sum_{x_i} p_X(x_i)$$

CORRELATION

- Measure of relationship between two RV
  - $m_{11} = E[XY]$
  - Measure away from independence (statistical)
- If $E[XY] = 0$, then X and Y are orthogonal
  - Note: orthogonal does not mean independent
  - Think of an inner product in RV space → 90 degree angle vs. statistical independence

- Note: "correlation does not imply causation"
  - Just because two variables are correlated, does not mean that one causes the other
  - E.g. increase in ice cream sales correlated with increase shark attacks. Probably not ice cream causing shark attacks but that ice cream and shark attacks happen more often during the summer

COVARIANCE

- $Cov(X, Y) = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)]$
  $= E[XY] - E[X]E[Y]$

- If $Cov(X, Y) = 0$ → X and Y uncorrelated
  - $E[XY] = E[X]E[Y]$
  - Note that independent RV are uncorrelated but uncorrelated does not imply independent

PEARSON'S CORRELATION COEFFICIENT

- Measure of **linear** dependence between X,Y
- $\rho(X,Y) = \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$
  - $|\rho_{XY}| \leq 1$

CONDITIONAL MEAN/VARIANCE

- Discrete
- Mean (expectation)
  - $\mu_{Y|x_i} = E[Y|x_i] = \sum_{y_j} y_j p_{Y|X}(y_j|x_i)$
- Variance
  - $\sigma^2_{Y|x_i} = Var(Y|x_i) = E\left[\left(Y - \mu_{Y|x_i}\right)^2 \middle| x_i\right]$

    $= \sum_{y_j} \left(y_j - \mu_{Y|x_i}\right)^2 p_{Y|X}(y_j|x_i)$

    $= E[Y^2|x_i] - E^2[Y|x_i]$
- Note: these values are a function of x_i and do not depend on Y
  - Defined for different x_i values

- Continuous
- Mean
  - $\mu_{Y|X} = E[Y|x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy$
- Variance
  - $\sigma^2_{Y|x_i} = Var(Y|x)$

    $= E\left[\left(Y - \mu_{Y|X}\right)^2 \middle| x\right]$

    $= \int_{-\infty}^{\infty} \left(y - \mu_{Y|x}\right)^2 f_{Y|X}(y|x)dy$

N-VARIATE RV

- Natural extension of bivariate discussion
- Give n-tuple of RVs $(X_1, X_2, \ldots, X_n)$ – n-dim random vector
  - Each X_i $i = 1,2, \ldots, n$ associates a real number to sample point $\xi \in S$
- We won't really work beyond bivariate in class
  - Ex: Joint CDF $F_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n)$

SPECIAL DISTRIBUTIONS

- Just like with single RV, there are important distributions that show up in nature a lot
  - Multinomial distribution – extension of binomial
  - N-variate Normal distribution

MULTINOMIAL DISTRIBUTION

- Multinomial trial (extension of binomial)
  - 1) Experiment with k possible outcomes that are mutually exclusive $(A_1, A_2, \dots, A_k)$
  - 2) $P(A_i) = p_i; \quad i = 1, \dots, k; \quad \sum_{i=1}^{k} p_i = 1$
- Multinomial RV
  - $(X_1, X_2, \dots, X_n)$ with X_i be RV denoting number of trials with result A_i
    - Count of number of each outcome
  - $p_{X_1 X_2 \dots X_k}(x_1, \dots, x_k) = \frac{n!}{x_1! x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$
    - Probability of combination of different outcomes

MULTINOMIAL EXAMPLE

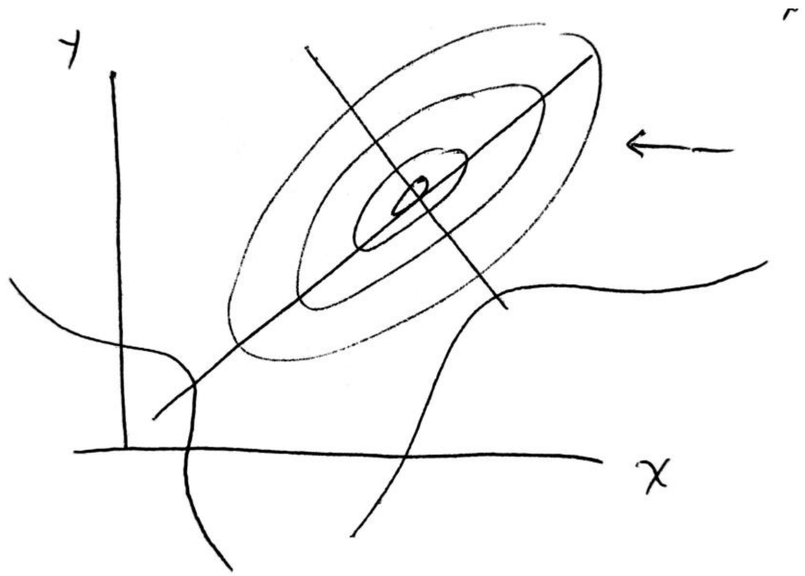
- k different color balls in a bag → p_i is the probability of color i to be drawn
- Select a ball at random and record the color then replace in bag
- Count of the colors at the end of the n ball draws is a multinomial RV
  - Distribution tells the probability of seeing e.g. 1 white, 2 red, 3 blue, and 4 green balls

NORMAL DISTRIBUTION

- Bivariate

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2}q(x, y)\right]$$

$$q(x, y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$$



- N-variate
  - Vector valued function (see book for details)

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\det K|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T K^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

  - Covariance matric

$$K = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix} \qquad \sigma_{ij} = \text{Cov}(X_i, X_j)$$

- Note: covariance controls shape or orientation in bivariate case