EE361: SIGNALS AND SYSTEMS II

CH1: PROBABILITY

http://www.ee.unlv.edu/~b1morris/ee361

SAMPLE SPACE AND EVENTS

CHAPTER 1.1-1.2

INTRODUCTION

- Very important mathematical framework that is used in every ECE discipline (and science and engineering in general)
- The world is not perfect and some variability occurs in everything we do professionally
  - The recorded sound of a voice saying a number multiple times
  - The pixel values of successive images

(RANDOM) EXPERIMENT

- Process of observation
  - (Random if outcome cannot be determined with certainty)
- Outcome – the result of an observation of an experiment
  - Will use ξ (xi) for an outcome
- Think of Schrödinger's cat paradox from quantum mechanics → an experiment has many possible outcomes but only after observation can it be known

EXAMPLE RANDOM EXPERIMENTS

- Roll of a dice
- Toss of a coin
- Drawing of a card from a deck

SAMPLE SPACE

- The universal set of sample space S – the set of all possible outcomes of a random experiment
- Sample point – an element in set S (a possible outcome)
- Cardinality – the number of elements in a set

EXAMPLES

- Single coin toss
  - $S = \{H, T\} = \{\text{Heads}, \text{Tails}\}$
  - $|S| = 2$ for two possible outcomes
- Two coin tosses
  - $S = \{HH, HT, TH, TT\}$
  - $|S| = 4$

SAMPLE SPACE

- Discrete S – finite number of sample points (outcomes) or countably infinite
  - Countable set – elements can be placed in one-to-one correspondence with positive integers
- Continuous S – sample points constitute a continuum
  - Example: transistor lifetime in hours
  - Transistor can live until it dies at time τ
  - $S = \{\tau : 0 \leq \tau < \infty\}$

EVENTS

- Set notations
- $\xi \in X$ - ξ is an element of (belongs to) set S
- $\xi \notin S$ - ξ is not an element of S
- $A \subset B$ – A contained in B
  - A is a subset of B if every element of A is contained in B
- Event – subset of sample space S
  - Even roll of die $A = \{2, 4, 6\}$
  - Sum of rolls greater than 6
- Elementary event – sample point of S (single outcome of S)
- Certain event – $S \subset S$

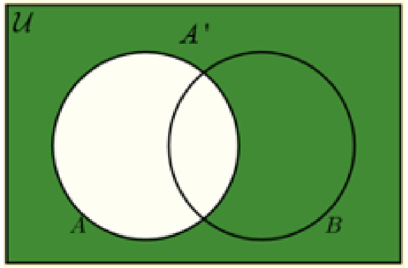
ALGEBRA OF SETS

CHAPTER 1.3

SET ALGEBRA I

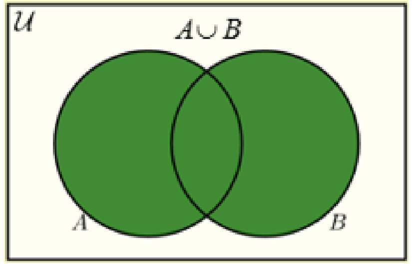
- 1 Equality – $A = B$ if $A \subset B$ and $B \subset A$
- 2 Complement – suppose $A \subset S$,
  - $\bar{A} = \{\xi : \xi \in S \text{ and } \xi \notin A\}$
    - All elements in S but not in A
    - $\bar{A}$ is event A did not occur
- 3 Union – set containing all elements in either A or B or both
  - $A \cup B = \{\xi : \xi \in A \text{ or } \xi \in B\}$
    - Event either A of B occurred
- 4 Intersection – set containing all elements in both A and B
  - $A \cap B = \{\xi : \xi \in A \text{ and } \xi \in B\}$
    - Both event A and B occurred

Venn Diagram



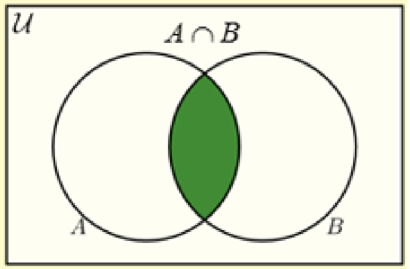
A complement

Elements that don't belong to *A*.



A union B

Elements that belong to either *A* or *B* or both.



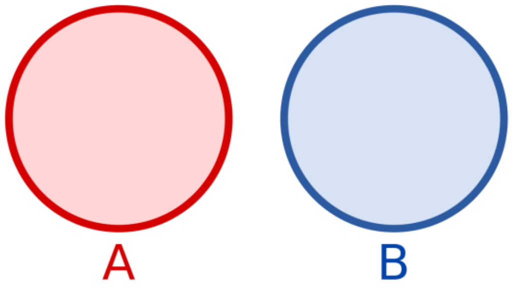
A intersect B

Elements that belong to both *A* and *B*.

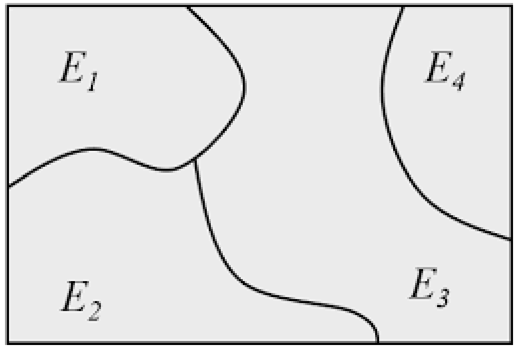
https://www.onlinemathlearning.com/venn-diagram.html

SET ALGEBRA II

- Null set – set containing no elements
  - $S = \phi$ or $S = \{\phi\}$
- Disjoint sets (mutually exclusive)
  - Sets A and B have no common elements → $A \cap B = \phi$



- Partition of S – a way to divide space S up completely
  - If $A_i \cap A_j = \phi$ $(i \neq j)$ and $\bigcup_{i=1}^{k} A_i = S$
  - Then $\{A_i : 1 \leq i \leq k\}$ is a partition



SET ALGEBRA III

- Size (cardinality) of set
  - $|A|$ - number of elements in A (when countable)
- Properties
  - If $A \cap B = \phi$, then $|A \cup B| = |A| + |B|$
  - $|\phi| = 0$
  - If $A \subset B$, then $|A| \leq |B|$
  - $|A \cup B| + |A \cap B| = |A| + |B|$
- (Cartesian) product of sets
  - $C = A \times B = \{(a, b): a \in A, b \in B\}$
  - Set of ordered pairs of elements
- Example
- $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2\}$
- $C = A \times B =$
  $\{(a_1, b_1), (a_1, b_2), (a_2, b_1),$
  $(a_2, b_2), (a_3, b_1), (a_3, b_2)\}$
- $D = B \times A =$
  $\{(b_1, a_1), (b_1, a_2), (b_1, a_3),$
  $(b_2, a_1), (b_2, a_2), (b_2, a_3)\}$

LAWS OF SETS

- Commutative
  - $A \cup B = B \cup A \qquad A \cap B = B \cap A$
- Associativity
  - $A \cup (B \cup C) = (A \cup B) \cup C \qquad A \cap (B \cap C) = (A \cap B) \cap C$
- Distributivity
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- De Morgan's
  - $\overline{A \cup B} = \bar{A} \cap \bar{B} \qquad \overline{A \cap B} = \bar{A} \cup \bar{B}$

EVENT SPACE

- Collection F of subsets of sample space S
  - Also known as a sigma field
- A set of subsets A such that
  - $S \in F$
  - If $A \in F$, then $\bar{A} \in F$
  - If $A_i \in F$ for $i \geq 1$, then $\cup_{i=1}^{\infty} A_i \in F$
- Example coin toss – $S = \{H, T\}$
  - $F_1 = \{S, \phi\}$, $\quad F_2 = \{S, \phi, H, T\}$
  - $F_3 = \{S, \phi, H\}$ – not event space since $\bar{H} = T$ not included

PROBABILITY SPACE/EQUALLY LIKELY EVENTS

CHAPTER 1.4-1.5

PROBABILITY SPACE

- Assigns a real number to events in event space F
  - Known as probability measure
- Given a random experiment with sample space S
  - A is an event defined in F
  - Probability of event A → $P(A)$
- Probability space is defined over an event space
  - Triple: (S, F, P) = (sample space, event space, probability measure)

PROBABILITY MEASURE

- Two methods to define
  - Classical definition
  - Relative frequency definition

CLASSICAL (FINITE OUTCOMES) DEFINITION

- Defined a priori, without need for experimentation
  - Only possible to use for simple problems with finite and equally likely outcomes
- $P(A) = \frac{|A|}{|S|}$ - probability of event A
- $P(\bar{A}) = \frac{|\bar{A}|}{|S|} = \frac{|S|-|A|}{|s|} = 1 - \frac{|A|}{|S|} = 1 - P(A)$
- Given disjoint sets $A \cap B = \phi$
  - $P(A \cup B) = P(A) + P(B)$

EXAMPLE: DICE ROLL

- $S = \{1, 2, 3, 4, 5, 6\}$
- Define events
  - A: Roll (outcome) is even $A = \{2,4,6\}$
  - B: Roll odd $B = \{1,3,5\}$
  - C: Roll prime $C = \{1,2,3,5\}$
- Probability of events
  - $P(A) = \frac{3}{6} = \frac{1}{2}$ $P(B) = \frac{1}{2}$ $P(C) = \frac{4}{6} = \frac{2}{3}$

RELATIVE FREQUENCY DEFINITION

- Repeated experiment definition
- Random experiment repeated n times
  - $P(A) = \lim_{n\to\infty} \frac{n(A)}{n}$ relative frequency of event A
    - $n(A)$ – number of times event A occurs
  - $0 \le \frac{n(A)}{n} \le 1$
    - 0 – never occurs 1 – occurs every time
- Like classical, $A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$

AXIOMATIC DEFINITION

- Given probability space (S, F, P) and event $A \in F$
- Axiom 1: $P(A) \geq 0$
- Axiom 2: $P(S) = 1$
  - One outcome certainly happens
- Axiom 3: $P(A \cup B) = P(A) + P(B)$, if $A \cap B = \phi$

ELEMENTARY PROPERTIES OF PROB.

- 1 $P(\bar{A}) = 1 - P(A)$
- 2 $P(\phi) = 0$
- 3 $P(A) \leq P(B)$ if $A \subset B$
- 4 $P(A) \leq 1 \Rightarrow 0 \leq P(A) \leq 1$
- 5 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $\leq P(A) + P(B)$
- 8 Given $A_1, A_2, \ldots, A_n$ finite sequence of mutually exclusive events $(A_i \cap A_j = \phi, i \neq j)$,
  $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

EQUALLY LIKELY EVENTS

CHAPTER 1.5

EQUALLY LIKELY EVENTS

- Finite sample space (n-finite)
  - $S = \{\xi_1, \xi_2, \dots, \xi_n\}$ and $P(\xi_i) = p_i$
    - $0 \le p_i \le 1 \qquad \sum_{i=1}^{n} p_i = 1$
    - $A = \bigcup_{i \in I} \xi_i \;\Rightarrow\; \sum_{\xi_i \in A} p(\xi_i) = \sum_{i \in I} p_i$

- Equally likely events
  - $p_1 = p_2 = \cdots = p_n$
  - $p_i = \frac{1}{n}, i = 1, 2, \dots, n$ and $P(A) = n(A)/n$

CONDITIONAL PROBABILITY

CHAPTER 1.6

CONDITIONAL PROBABILITY

- Probability of an event A given event B has occurred
  - $P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$
    - $P(A \cap B)$ – joint probability of A and B
  - $P(B|A) = \frac{P(A \cap B)}{P(A)}$
  - $\Rightarrow P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

BAYES RULE

- Incredibly important relationship from joint probability

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

  - $P(A|B)$ - posterior (what we want to estimate)
  - $P(B|A)$ – likelihood (probability of observing B given A)
  - $P(A)$ – prior (probability of A with no extra information)
  - $P(B)$ – marginal likelihood (total probability)
- Tells how to update our believes based on the arrival of new, relevant pieces of evidence

TOTAL PROBABILITY/INDEPENDENT EVENTS

CHAPTER 1.7-1.8

TOTAL PROBABILITY

- Let B be an event in S
  - $P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$
  - Exhaustive
    - $\bigcup_{i=1}^{n} A_i = S$
  - Mutually exclusive
    - $A_i \cap A_j = \phi$
- From Bayes
  - $P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$

INDEPENDENT EVENTS

- Two events are statistically independent iff
- $P(A \cap B) = P(A)P(B)$
- $\Rightarrow P(A|B) = P(A)$ and $P(B|A) = P(B)$
- If independent, probability of event A is not changed by knowledge of event B
- Note:
  - Mutually exclusive $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$
  - Independent $P(\cap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i)$