EE361: SIGNALS AND SYSTEMS II
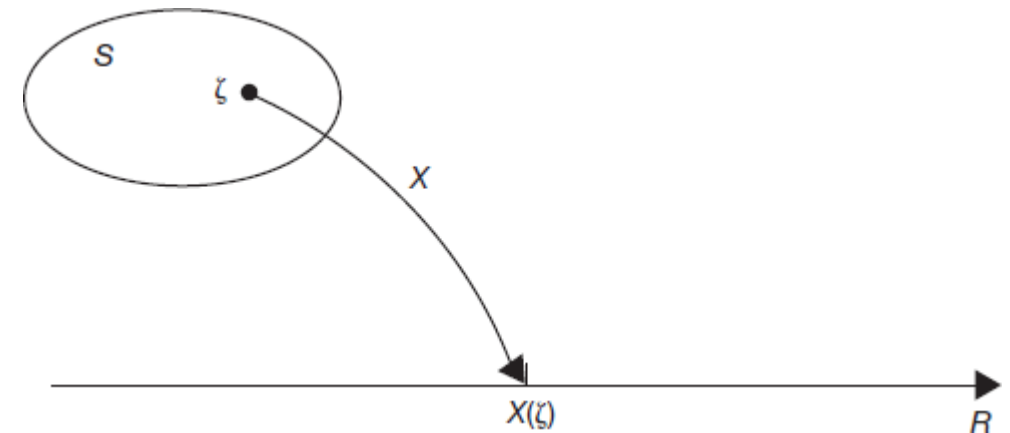
CH2: RANDOM VARIABLES

INTRODUCTION

- A Random Variable is a function that maps an event to a probability (real value)

- Will use distribution functions to describe the functional mapping

  - Example: your score on the midterm is a random variable and the Gaussian distribution explains the probability you achieved a certain value (e.g. 70/100)

RANDOM VARIABLE

- $X(\xi)$ is a single-valued real function that assigns a real number (value) to each sample point (outcome) in a sample space S

  - Often just use X for simplicity

  - This is a function (mapping) from sample space S (domain of X) to values (range)

  - This is a many-to-one mapping

    - Different ξ_i may have same value $X(\xi_i)$, but two values cannot come from same outcome

EVENTS DEFINED BY RVS

- Event
  - $(X = x) = \{\xi : X(\xi) = x\}$
  - RV X value is x, a fixed real number
- Similarly,
  - $(x_1 < X \leq X_2) = \{\xi : x_1 < X(\xi) \leq x_2\}$
- Probability of event
  - $P(X = x) = P\{\xi : X(\xi) = x\}$
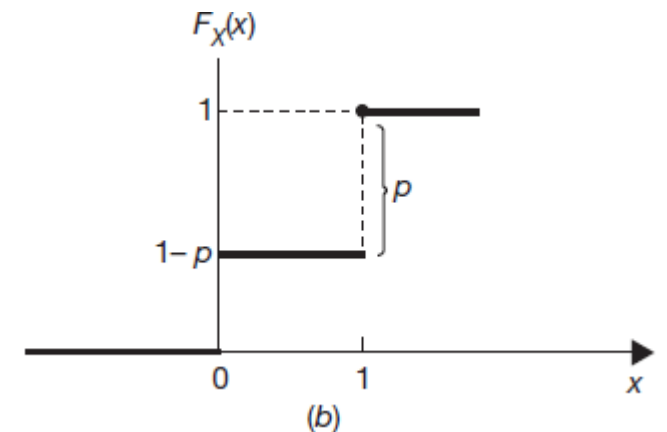
EXAMPLE: COIN TOSS 3 TIMES

- Sample space $S = \{HHH, HHT, \ldots, TTT\}$, $|S| = 2^3 = 8$
- Define RV X as the number of heads after the three tosses
- Find $P(X = 2)$
  - Event A: $(X = 2) = \{\xi : X(\xi) = 2\} = \{\text{HHT}, \text{HTH}, \text{HTT}\}$
  - By equally likely events
    - $P(A) = P(X = 2) = \frac{|A|}{|S|} = \frac{3}{8}$
- Find $P(X < 2)$
  - Event B: $(X < 2) = \{\xi : X(\xi) < 2\} = \{\text{HTT}, \text{THT}, \text{HTT}, \text{TTT}\}$ (1 or less heads)
  - By equally likely events
    - $P(B) = P(X < 2) = \frac{|B|}{|S|} = \frac{4}{8} = \frac{1}{2}$

CUMULATIVE DISTRIBUTION FUNCTION (CDF)

- $F_X(x) = P(X \leq x)$ $-\infty < x < \infty$
  - F – the CDF
  - X – the RV of interest
  - x – the value the RV will take

- Note: this is an increasing (non-decreasing) function

CDF PROPERTIES

- 1) $0 \leq F_X(x) \leq 1$
  - Must be less than some maximal value

- 2) $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$
  - Non-decreasing function

- ...

- 5) $\lim_{x \to a^+} F_X(x) = F_X(a^+) = F_X(x)$
  with $a^+ = \lim_{0 < \epsilon \to 0} a + \epsilon$
  - Continuous from the right

$F_X(x)$

1

$1-p$

p

0 1 x

(b)

EXAMPLE: 3 COIN TOSS AGAIN

- X – number of heads in three tosses

x (value)	Event ($X \leq x$)	# elements	$F_X(x)$
-1	∅	0	0
0	{TTT}	1 (1 + 0)	$\dfrac{1}{8}$
1			
2			
3			
4			

EXAMPLE: 3 COIN TOSS AGAIN

- X – number of heads in three tosses

x (value)	Event ($X \leq x$)	# elements	$F_X(x)$
-1	$\emptyset$	0	0
0	{TTT}	**1** (1 + 0)	$\dfrac{1}{8}$
1	{HTT, THT, TTH, **TTT**}	4 (3+**1**)	$\dfrac{4}{8} = \dfrac{1}{2}$
2			
3			
4			

EXAMPLE: 3 COIN TOSS AGAIN

- X – number of heads in three tosses

x (value)	Event ($X \leq x$)	# elements	$F_X(x)$
-1	∅	0	0
0	{TTT}	1 (1 + 0)	$\frac{1}{8}$
1	{HTT, THT, TTH, TTT}	4 (3+1)	$\frac{4}{8} = \frac{1}{2}$
2	{HHT, HTH, THH, **HTT, THT, TTH, TTT**}	7 (3 + 4)	$\frac{7}{8}$
3			
4			

EXAMPLE: 3 COIN TOSS AGAIN

- X – number of heads in three tosses

x (value)	Event ($X \leq x$)	# elements	$F_X(x)$
-1	$\emptyset$	0	0
0	{TTT}	1 (1 + 0)	$\dfrac{1}{8}$
1	{HTT, THT, TTH, TTT}	4 (3+1)	$\dfrac{4}{8} = \dfrac{1}{2}$
2	{HHT, HTH, THH, HTT, THT, TTH, TTT}	**7** (3 + 4)	$\dfrac{7}{8}$
3	{HHH, **HHT, HTH, THH, HTT, THT, TTH, TTT**}	8 (1 + **7**)	1
4	S	8 (0 + 8)	1

EXAMPLE: 3 COIN TOSS AGAIN

- X – number of heads in three tosses

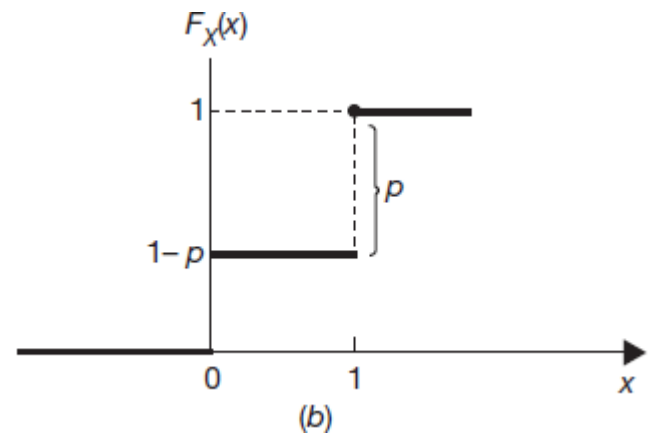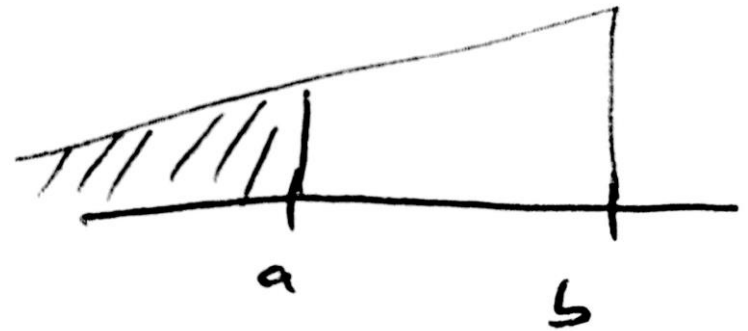| x (value)	Event ($X \leq x$)	# elements	$F_X(x)$
-1	$\emptyset$	0	0
0	{TTT}	1	$\dfrac{1}{8}$
1	{HTT, THT, TTH, TTT}	4 (3+1)	$\dfrac{4}{8} = \dfrac{1}{2}$
2	{HHT, HTH, THH, HTT, THT, TTH, TTT}	7 (3 + 4)	$\dfrac{7}{8}$
3	{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}	8 (1 + 7)	1
4	S	8 (0 + 8)	1

PROBABILITIES FROM CDF
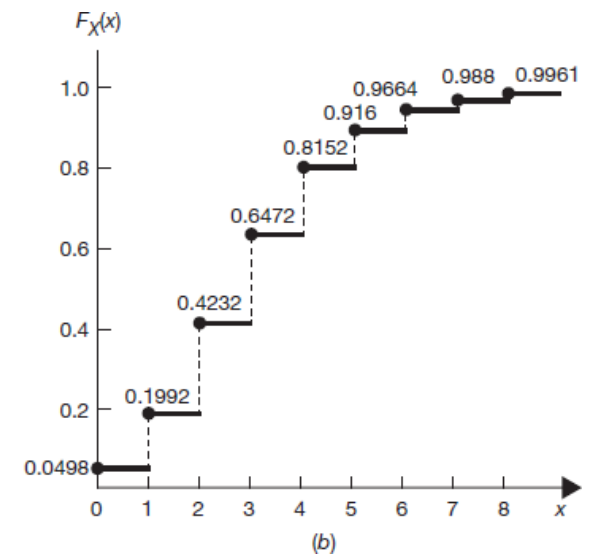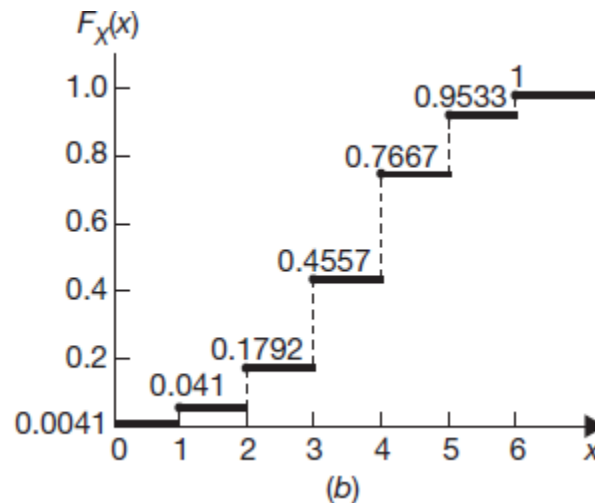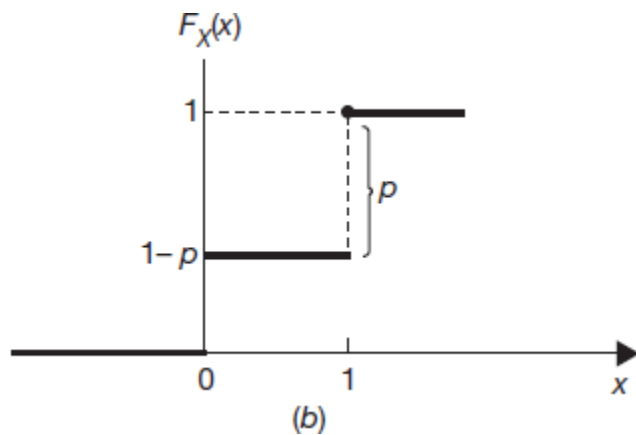
- Completely specify probabilities from a CDF

- 1) $P(a < X \leq b) = F_X(b) - F_X(a)$
  $$= P(X \leq b) - P(X \leq a)$$

- 2) $P(X > a) = 1 - F_X(a)$

- 3) $P(X < b) = F_X(b^-)$
  - $b^- = \lim_{0 < \epsilon \to 0} b - \epsilon$
  - Approach from the left side

DISCRETE RV

- *X* is RV with CDF $F_X(x)$ and $F_X(x)$ only changes in jumps (countably many) and is constant between jumps

  - Range of *X* contains a finite (countably infinite) number of points

PROBABILITY MASS FUNCTION (PMF)

- Given jumps in discrete RV @ points $x_1, x_2, \ldots$ and $x_i < x_j$ for $i < j$

  - $p_X(x) = F_X(x_i) - F_X(x_{i-1})$
    $$= P(X \leq x_i) - P(X \leq x_{i-1}) = P(X = x_i)$$

- 3 Coin toss example

x (value)	# elements	$F_X(x)$	$p_X(x)$	Discussion
1	4 (3+1)	$\frac{4}{8} = \frac{1}{2}$	$p_X(1) = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$	\<how much more needed from previous value\>
2	7 (3 + 4)	$\frac{7}{8}$	$p_X(2) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$	3 extra outcomes
3	8 (1 + 7)	1	$p_X(3) = 1 - \frac{7}{8} = \frac{1}{8}$	1 extra outcome

PMF PROPERTIES

- 1) $0 \leq p_X(x_k) \leq 1 \quad k = 1, 2, \ldots$ (finite set of values)
- 2) $p_X(x) = 0$ if $x \neq x_k$ (a value that cannot occur)
- 3) $\sum_k p_X(x_k) = 1$

- CDF from PMF
  - $F_X(x) = P(X \leq x) = \sum_{x_k \leq x} p_X(x_k)$
  - Accumulation of probability mass

CONTINUOUS RV

- X is RV with CDF $F_X(x)$ continuous and has a derivative $\frac{dF_X(x)}{dx}$ exists
  - Range contains an interval of real numbers

- Note: $P(X = x) = 0$
  - There is zero probability for a particular continuous outcome $\rightarrow$ only over a range of values

PROBABILITY DENSITY FUNCTION (PDF)

- $f_X(x) = \frac{dF_X(x)}{dx}$ pdf of X

- Properties

- 1) $f_X(x) \geq 0$

- 2) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

- 3) $f_X(x)$ is piecewise continuous

- 4) $P(a < X \leq b) = \int_a^b f_X(x)dx$
  $$= P(a \leq X \leq b)$$
  $$= F_X(b) - F_X(a)$$

- CDF from PDF

  - $F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(\xi)d\xi$

MEAN

- Expected value of RV X

- Discrete

  - $\mu_X = E[X] = \sum_k x_k p_X(x_k)$

- Continuous

  - $\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$

MOMENT

- n^{th} moment defined as

- Discrete

  - $E[X^n] = \sum_k x_k^n P_X(x_k)$

- Continuous

  - $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$

VARIANCE

- $\sigma_X^2 = Var(X) = E[(X - E[X])^2]$
  - $E[.]$ – expected value operation
  - $E[X] = \mu_X$ - mean
- Discrete
  - $\sigma_X^2 = \sum_k (x - \mu_X)^2 p_X(x_k)$
- Continuous
  - $\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$

$$Var(X) = E\left[(X - E[X])^2\right]$$
$$= E[X^2 - 2X\mu_X + \mu_X^2]$$
$$= E[X^2] - 2\mu_X E[X] + \mu_X^2$$
$$= E[X^2] - 2\mu_X^2 + \mu_X^2$$
$$= E[X^2] - \mu_X^2$$
$$= \underbrace{E[X^2]}_{\text{2nd moment}} - \underbrace{E^2[X]}_{\text{1st moment}}$$

IMPORTANT DISTRIBUTIONS

- Model real-world phenomena

- Mathematically convenient specification for probability distribution (usually pmf or pdf)

- Will examine similar discrete and continuous distributions

  - Note: will leave most of content for the book rather than in slides