DISCRETE TIME FOURIER SERIES

CHAPTER 3.6

DTFS VS CTFS DIFFERENCES

- While quite similar to the CT case,
  - DTFS is a finite series, $\{a_k\}, |k| < K$
  - Does not have convergence issues

- Good News: motivation and intuition from CT applies for DT case

DTFS TRANSFORM PAIR

- Consider the discrete time periodic signal $x[n] = x[n+N]$

- $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$ synthesis equation
- $a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$ analysis equation

- N – fundamental period (smallest value such that periodicity constraint holds)
- $\omega_0 = 2\pi/N$ – fundamental frequency
- $\sum_{n=<N>}$ indicates summation over a period (N samples)

DTFS REMARKS

- DTFS representation is a finite sum, so there is always pointwise convergence
- FS coefficients are periodic with period N

DTFS PROOF

- Proof for the DTFS pair is similar to the CT case
- Relies on orthogonality of harmonically related DT period complex exponentials
- Will not show in class

HOW TO FIND DTFS REPRESENTATION

- Like CTFS, will use important examples to demonstrate common techniques

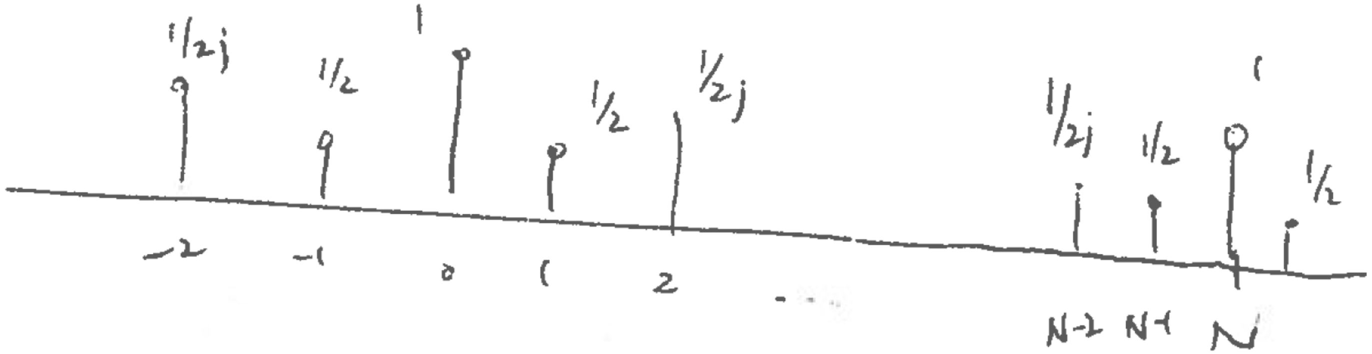
- Sinusoidal signals – Euler's relationship
- Direct FS summation evaluation – periodic rectangular wave and impulse train
- FS properties table and transform pairs

SINUSOIDAL SIGNAL

- $x[n] = 1 + \frac{1}{2}\cos\left(\frac{2\pi}{N}\right)n + \sin\left(\frac{4\pi}{N}\right)n$
- First find the period
- Rewrite $x[n]$ using Euler's and read off a_k coefficients by inspection
- Shortcut here

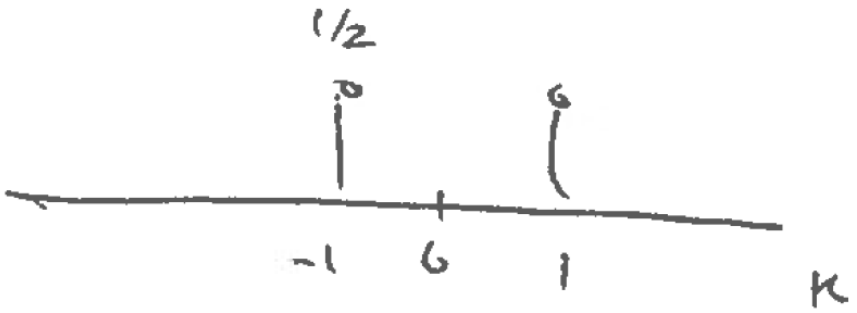
$$
\begin{aligned}
x[n] &= 1 + \frac{1}{2}\cos\left(\frac{2\pi}{N}\right)n + \sin\left(\frac{4\pi}{N}\right)n \\
&= 1 + \frac{1}{4}\left(e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right) + \frac{1}{2j}\left(e^{j\frac{4\pi}{N}n} - e^{-j\frac{4\pi}{N}n}\right) \\
&= 1 + \frac{1}{4}\left(e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right) + \frac{1}{2j}\left(e^{j2\frac{2\pi}{N}n} - e^{-j2\frac{2\pi}{N}n}\right)
\end{aligned}
$$

- $a_0 = 1, a_{\pm 1} = \frac{1}{4}, a_2 = a_{-2}^* = \frac{1}{2j}$

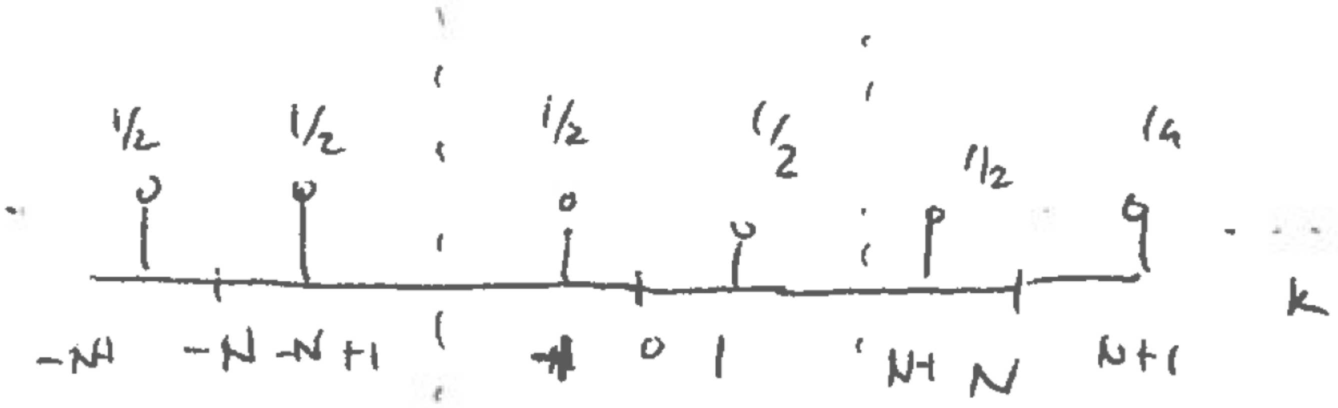


SINUSOIDAL COMPARISON

- $x(t) = \cos \omega_0 t$
- $a_k = \begin{cases} 1/2 & k = \pm 1 \\ 0 & else \end{cases}$



- $x[n] = \cos \omega_0 n$
- $a_k = \begin{cases} 1/2 & k = \pm 1 \\ 0 & else \end{cases}$
  - Over a single period → must specify period with period N



PERIODIC RECTANGLE WAVE

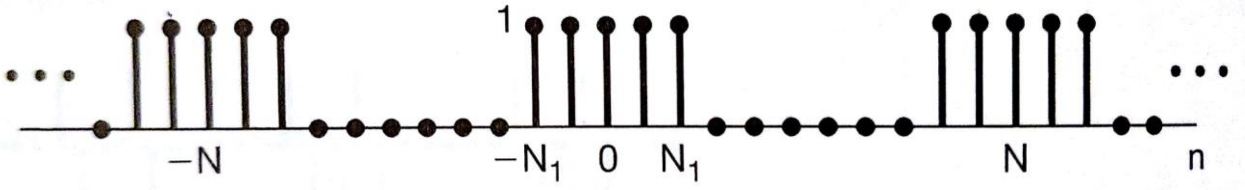


Figure 3.16 Discrete-time periodic square wave.

$$k = \begin{matrix} 0 \\ \pm N \\ \pm 2N \\ \vdots \end{matrix} \qquad a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1 = \frac{2N_1+1}{N}$$

$$x[n] = \begin{cases} 1 & |n| < N_1 \\ 0 & N_1 < |n| < N/2 \end{cases}$$

$$\updownarrow$$

$$a_k = \begin{cases} (2N_1+1)/N & k = 0, \pm N, \pm 2N, \ldots \\ \frac{\sin 2\pi k(N_1+1/2)/N}{\sin k\pi/N} & k \neq 0, \pm N, \pm 2N, \ldots \end{cases}$$

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n} \\ &= \frac{1}{N}\sum_{n=-N/2}^{N/2-1} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-N_1}^{N_1}\alpha^n \end{aligned}$$

Remember the truncated geometric series $\sum_{n=0}^{N-1}\alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{m=0}^{2N_1}\alpha^{m-N_1} \\ &= \frac{1}{N}\alpha^{-N_1}\sum_{m=0}^{2N_1}\alpha^m = \frac{1}{N}\alpha^{-N_1}\left(\frac{1-\alpha^{2N_1+1}}{1-\alpha}\right) \\ &= \frac{1}{N}e^{-jk\omega_0 N_1}\left(\frac{1-e^{jk\omega_0(2N_1+1)}}{1-e^{-jk\omega_0}}\right) \\ &= \ldots \\ &= \frac{\sin 2\pi k\left(N_1+\frac{1}{2}\right)/N}{\sin k\omega_0/2} = \frac{\sin 2\pi k(N_1+1/2)/N}{\sin k\pi/N} \end{aligned}$$

RECTANGLE WAVE COEFFICIENTS

- Consider different "duty cycle" for the rectangle wave
  - 50% (square wave)
  - 25%
  - 12.5%
- Note all plots are still a sinc shaped, but periodic
  - Difference is how the sync is sampled
  - Longer in time (larger N) smaller spacing in frequency → more samples between zero crossings

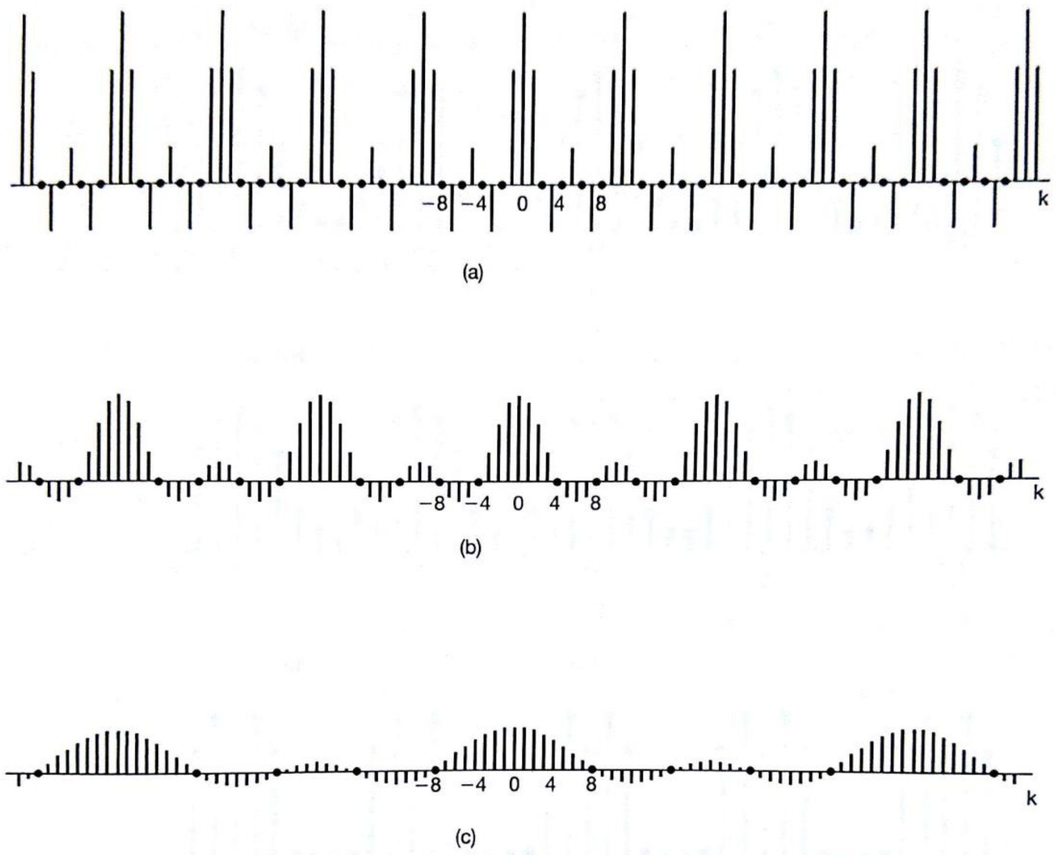
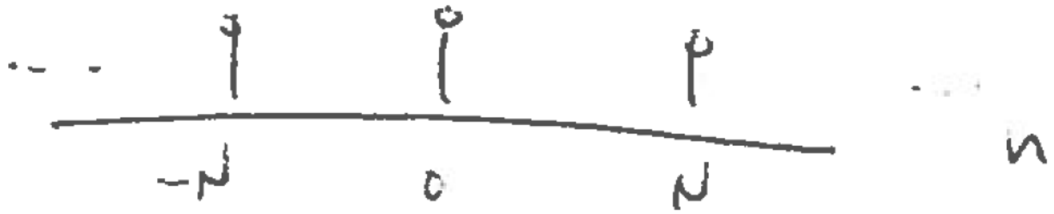


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

PERIODIC IMPULSE TRAIN

- $x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$
- Using FS integral



$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} dt$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum \delta[n - kN] e^{-jk\omega_0 n} dt$$

- Notice only one impulse in the interval

$$= \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-jk\omega_0 n} dt$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-jk\omega_0 0} dt = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] = \frac{1}{N}$$