CONTINUOUS TIME FOURIER SERIES

CHAPTER 3.3-3.8

CTFS TRANSFORM PAIR

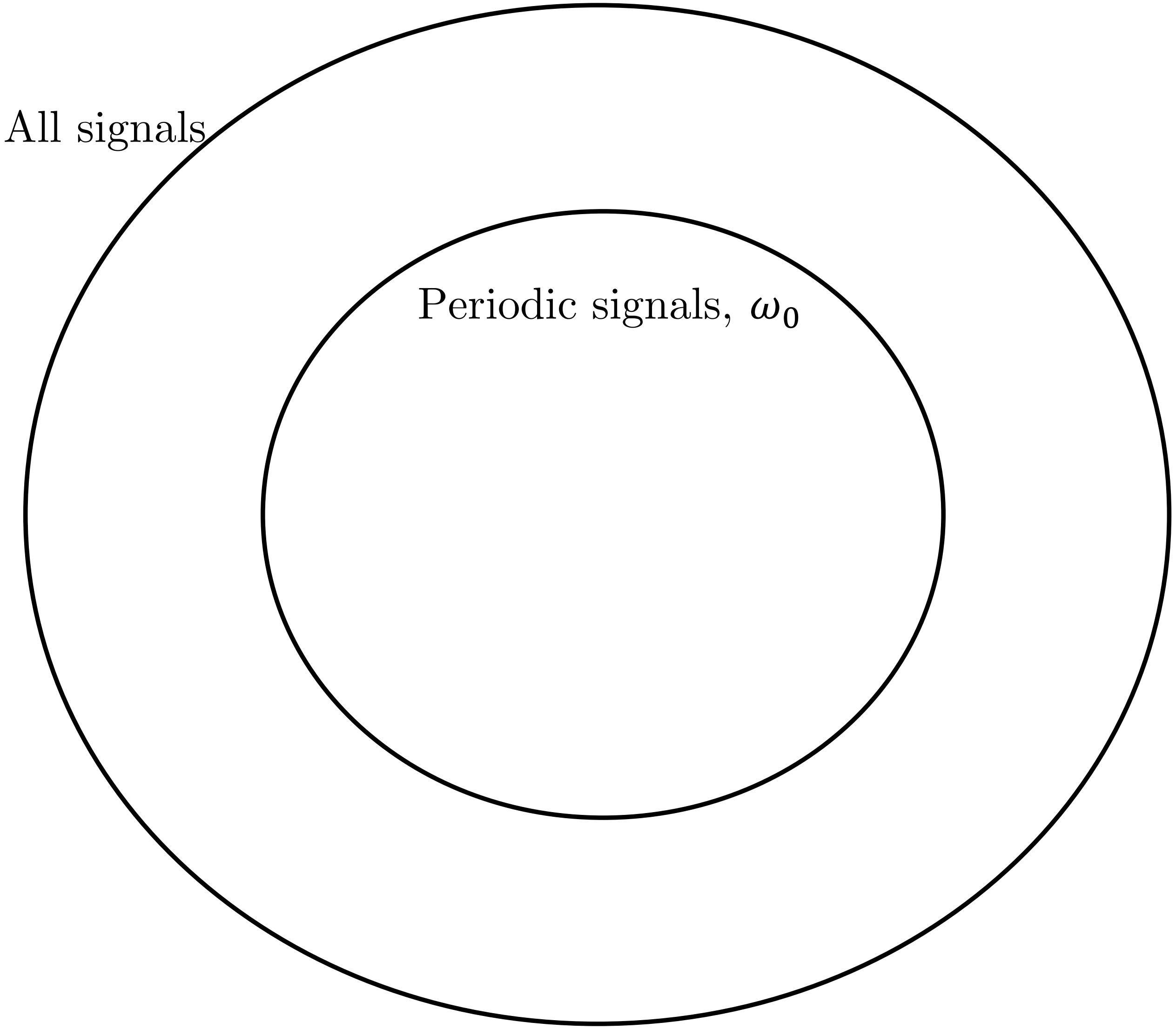
- Suppose $x(t)$ can be expressed as a linear combination of harmonic complex exponentials
  - $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ synthesis equation
- Then the FS coefficients $\{a_k\}$ can be found as
  - $a_k = \frac{1}{T} \int_T \ x(t)\, e^{-jk\omega_0 t} dt$ analysis equation
- ω_0 - fundamental frequency
- $T = 2\pi/\omega_0$ - fundamental period
- a_k known as FS coefficients or spectral coefficients

CTFS PROOF

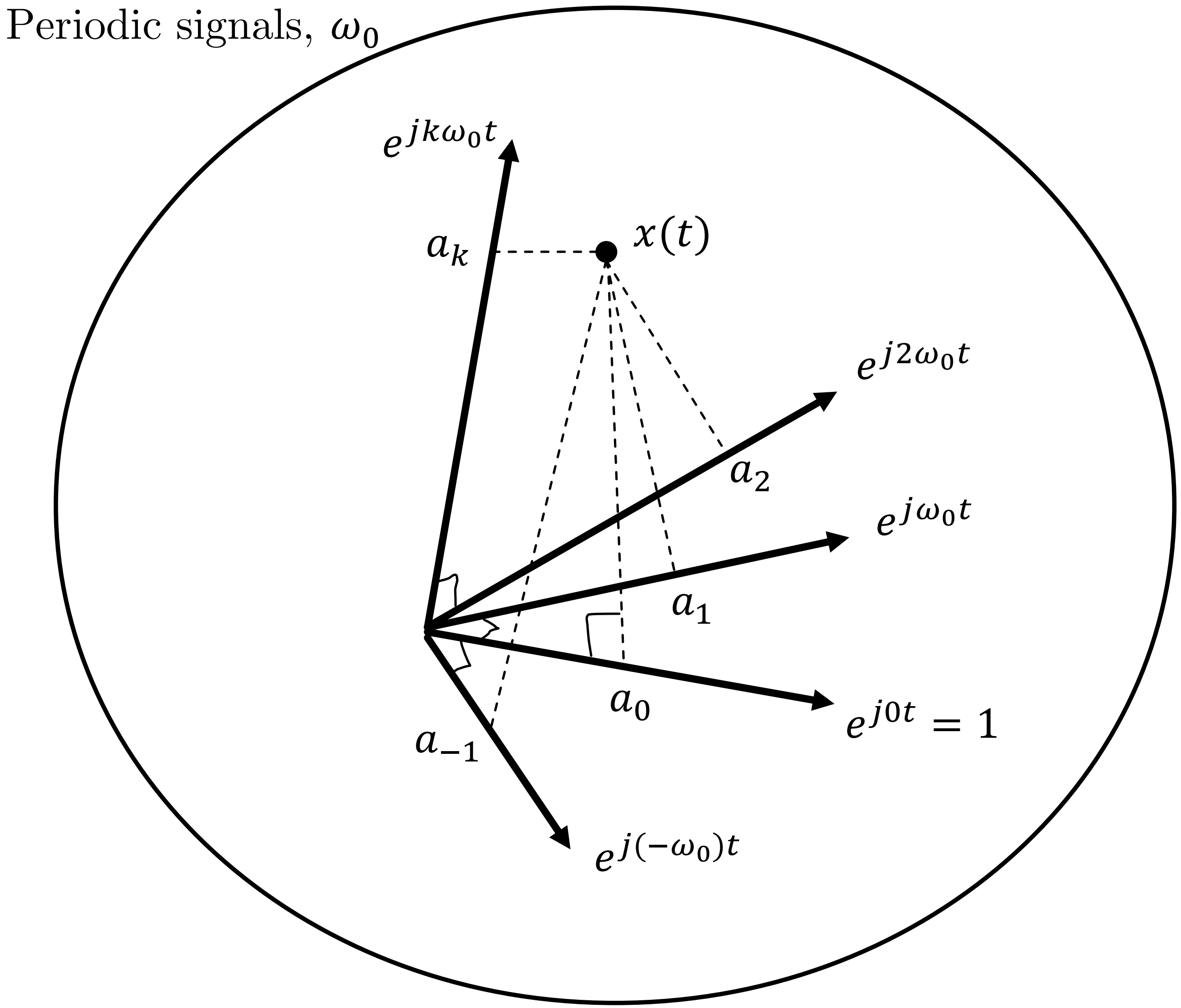
- While we can prove this, it is not well suited for slides.
  - See additional handout for details

- Key observation from proof: Complex exponentials are orthogonal

VECTOR SPACE OF PERIODIC SIGNALS



VECTOR SPACE OF PERIODIC SIGNALS



- Each of the harmonic exponentials are orthogonal to each other and span the space of periodic signals
- The projection of $x(t)$ onto a particular harmonic (a_k) gives the contribution of that complex exponential to building $x(t)$
  - a_k is how much of each harmonic is required to construct the periodic signal $x(t)$

HARMONICS

- $k = \pm 1 \Rightarrow$ fundamental component (first harmonic)
  - Frequency ω_0, period $T = 2\pi/\omega_0$
- $k = \pm 2 \Rightarrow$ second harmonic
  - Frequency $\omega_2 = 2\omega_0$, period $T_2 = T/2$ (half period)
- ...
- $k = \pm N \Rightarrow$ Nth harmonic
  - Frequency $\omega_N = N\omega_0$, period $T_N = T/N$ (1/N period)
- $k = 0 \Rightarrow a_0 = \frac{1}{T}\int_T x(t)dt$, DC, constant component, average over a single period

HOW TO FIND FS REPRESENTATION

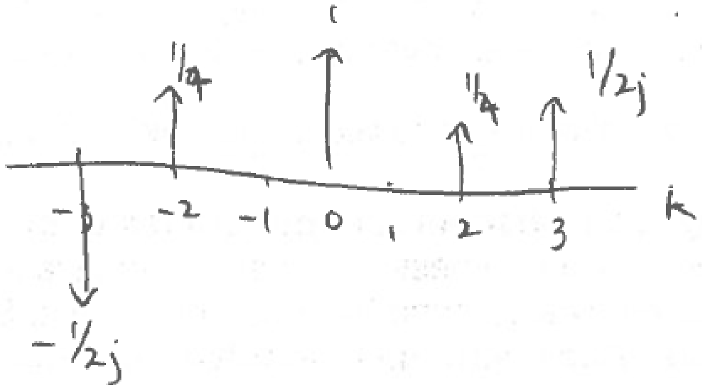
- Will use important examples to demonstrate common techniques

- Sinusoidal signals – Euler's relationship
- Direct FS integral evaluation
- FS properties table and transform pairs

SINUSOIDAL SIGNAL

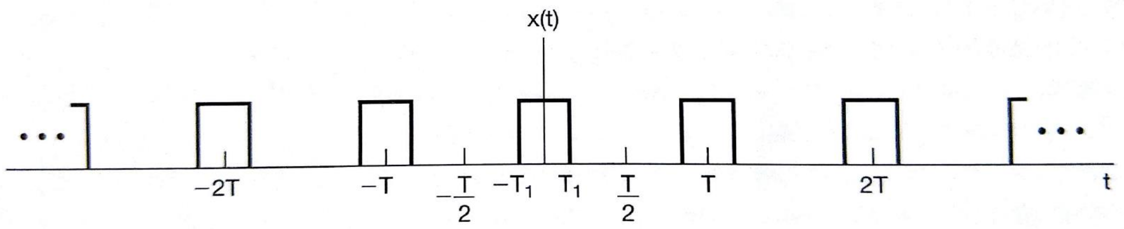
- $x(t) = 1 + \frac{1}{2}\cos 2\pi t + \sin 3\pi t$
- First find the period
  - Constant 1 has arbitrary period
  - $\cos 2\pi t$ has period $T_1 = 1$
  - $\sin 3\pi t$ has period $T_2 = 2/3$
  - $T = 2, \omega_0 = 2\pi/T = \pi$
- Rewrite $x(t)$ using Euler's and read off a_k coefficients by inspection

- $x(t) = 1 + \frac{1}{4}\left[e^{j2\omega_0 t} + e^{-j2\omega_0 t}\right] + \frac{1}{2j}\left[e^{j3\omega_0 t} - e^{-j3\omega_0 t}\right]$
- Read off coeff. directly
  - $a_0 = 1$
  - $a_1 = a_{-1} = 0$
  - $a_2 = a_{-2} = 1/4$
  - $a_3 = 1/2j$, $a_{-3} = -1/2j$
  - $a_k = 0$, else



PERIODIC RECTANGLE WAVE

- $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases}$



$k \neq 0$

$$a_k = \frac{1}{T}\int_T e^{-jk\omega_0 t}dt$$
$$= -\frac{1}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1} = \frac{1}{jk\omega_0 T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right]$$
$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$
$$= \underbrace{\frac{\sin(k\omega_0 T)}{k\pi}}_{\text{modulated sin function}} .$$

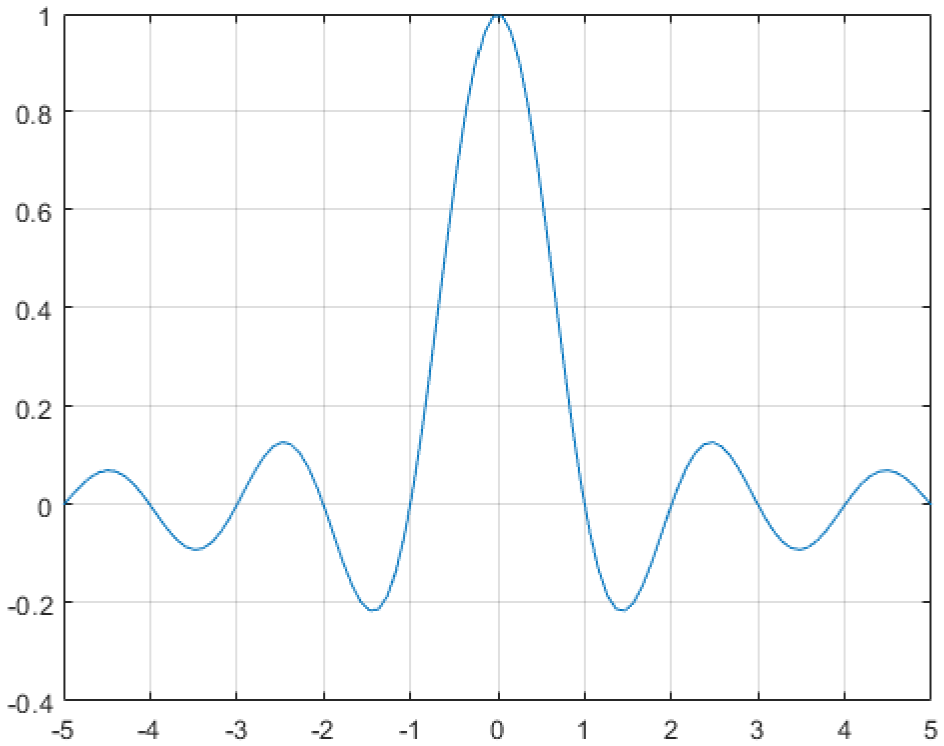
$k = 0$

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases} \longleftrightarrow a_k = \begin{cases} 2T_1/T & k = 0 \\ \frac{\sin(k\omega_0 T)}{k\pi} & k \neq 0 \end{cases}$$

SINC FUNCTION

- Important signal/function in DSP and communication
  - $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$ normalized
  - $\mathrm{sinc}(x) = \frac{\sin x}{x}$ unnormalized
- Modulated sine function
  - Amplitude follows 1/x
  - Must use L'Hopital's rule to get x=0 time



RECTANGLE WAVE COEFFICIENTS

- Consider different "duty cycle" for the rectangle wave
  - $T = 4T_1$ 50% (square wave)
  - $T = 8T_1$ 25%
  - $T = 16T_1$ 12.5%
- Note all plots are still a sinc shape
  - Difference is how the sync is sampled
  - Longer in time (larger T) smaller spacing in frequency → more samples between zero crossings

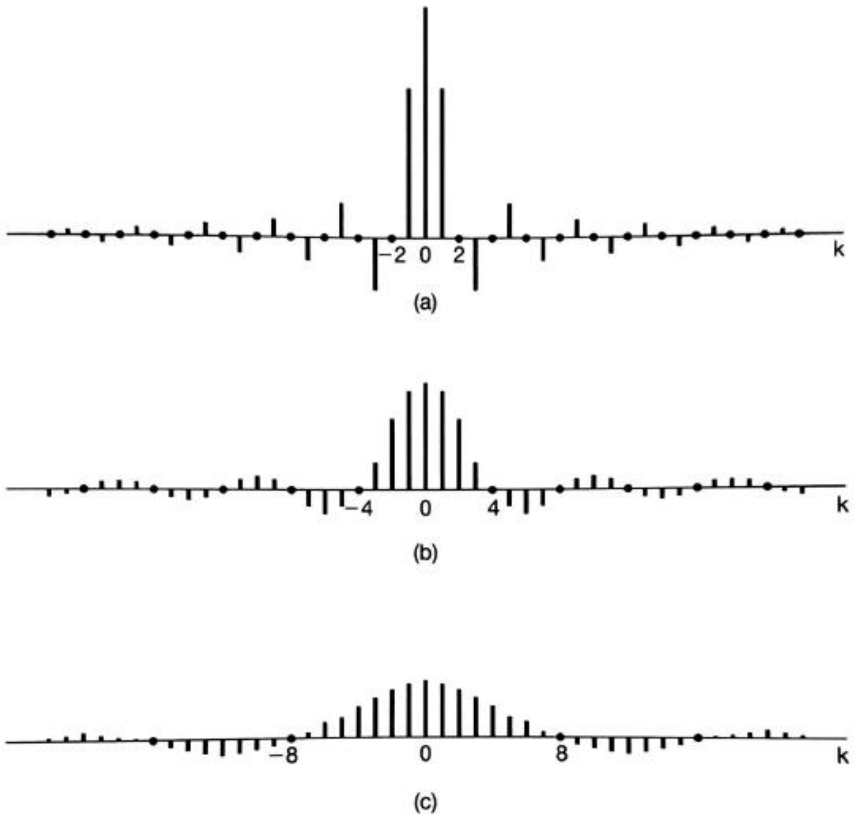
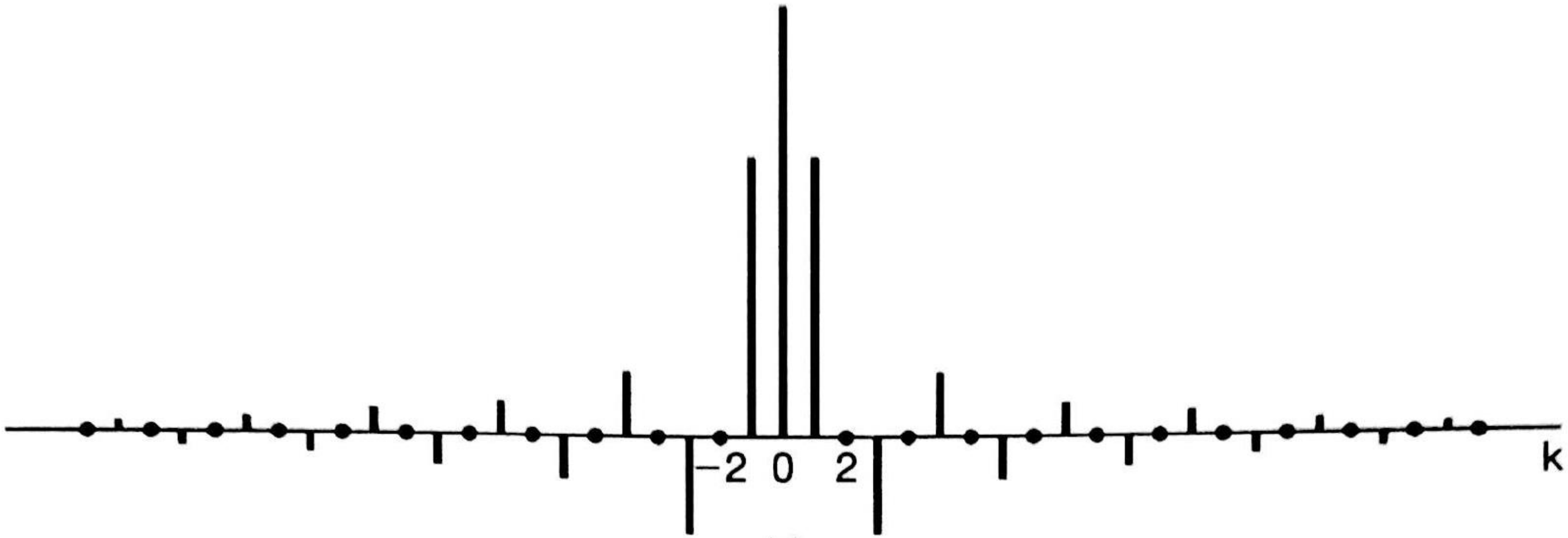


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

SQUARE WAVE

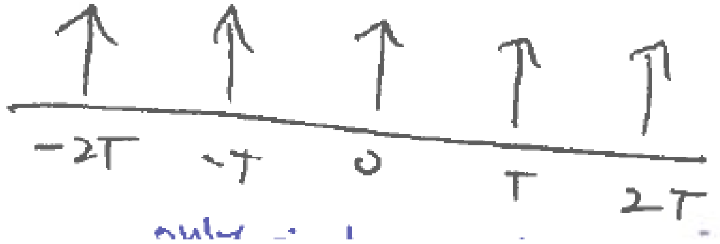
- Special case of rectangle wave with $T = 4T_1$
  - One sample between zero-crossing
- $a_k = \begin{cases} 1/2 & k = 0 \\ \dfrac{\sin(k\pi/2)}{k\pi} & else \end{cases}$



(a)

PERIODIC IMPULSE TRAIN

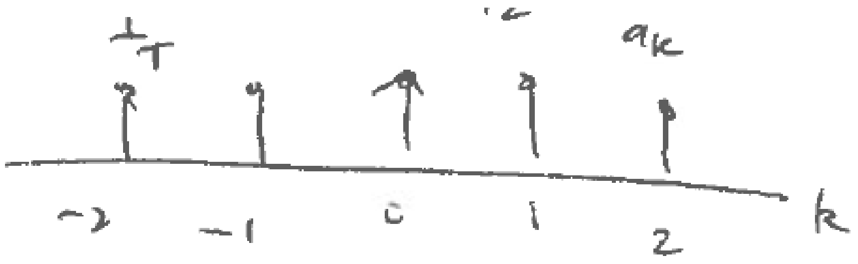
- $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$
- Using FS integral



$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$
$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum \delta(t - kT)e^{-jk\omega_0 t}dt$$

- Notice only one impulse in the interval

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt$$
$$a_k = \frac{1}{T}\underbrace{\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 0}\,dt}_{=1} = \frac{1}{T}$$



PROPERTIES OF CTFS

- Since these are very similar between CT and DT, will save until after DT

- Note: As for LT and Z Transform, properties are used to avoid direct evaluation of FS integral
  - Be sure to bookmark properties in Table 3.1 on page 206