EE361: SIGNALS AND SYSTEMS II

CH3: FOURIER SERIES

FOURIER SERIES OVERVIEW, MOTIVATION, AND HIGHLIGHTS

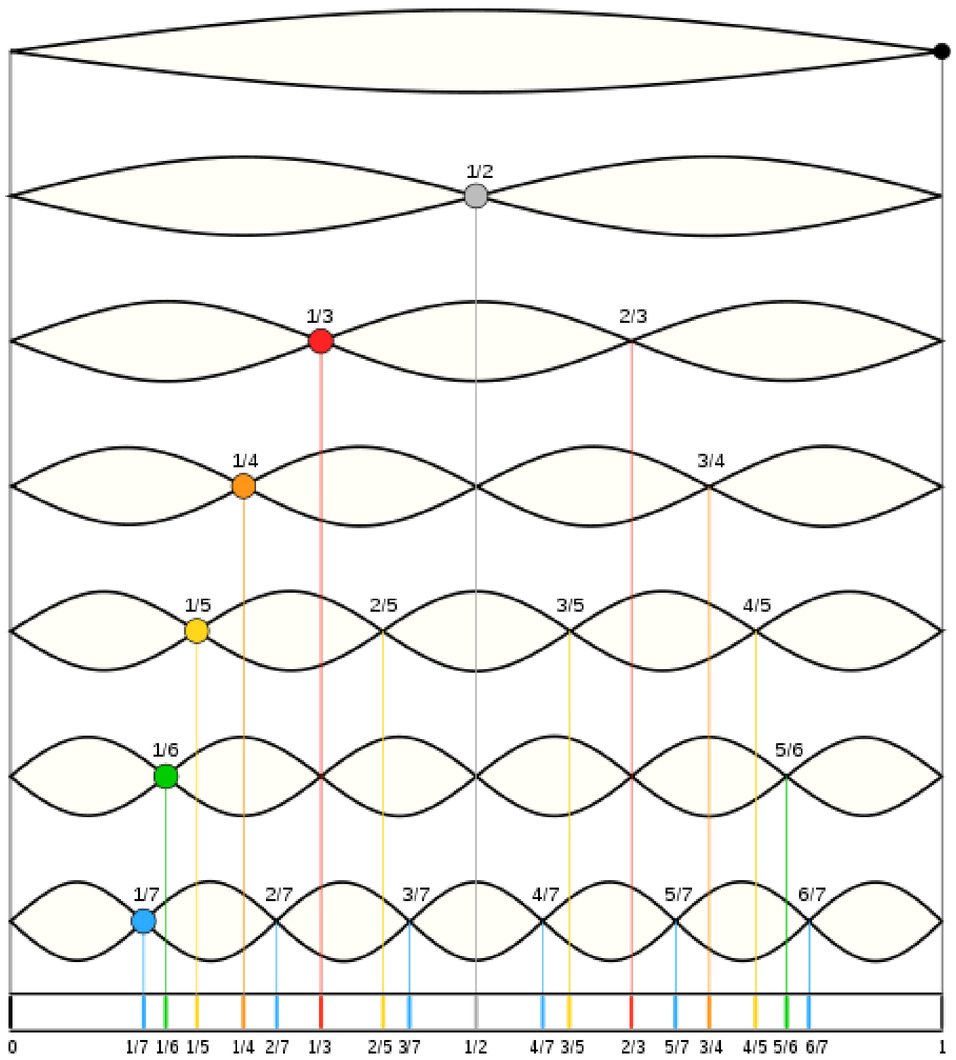
CHAPTER 3.1-3.2

BIG IDEA: TRANSFORM ANALYSIS

- Make use of properties of LTI system to simplify analysis
- Represent signals as a linear combination of basic signals with two properties
  - Simple response: easy to characterize LTI system response to basic signal
  - Representation power: the set of basic signals can be use to construct a broad/useful class of signals

NORMAL MODES OF VIBRATING STRING

- When plucking a string, length is divided into integer divisions or harmonics
  - Frequency of each harmonic is an integer multiple of a "fundamental frequency"
  - Also known as the normal modes
- Any string deflection could be built out of a linear combination of "modes"



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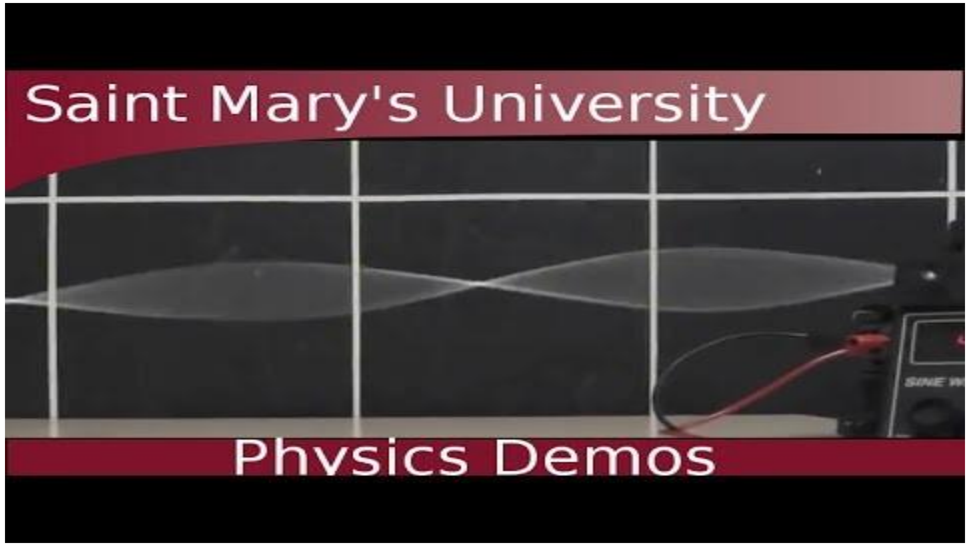


Caution: turn your sound down

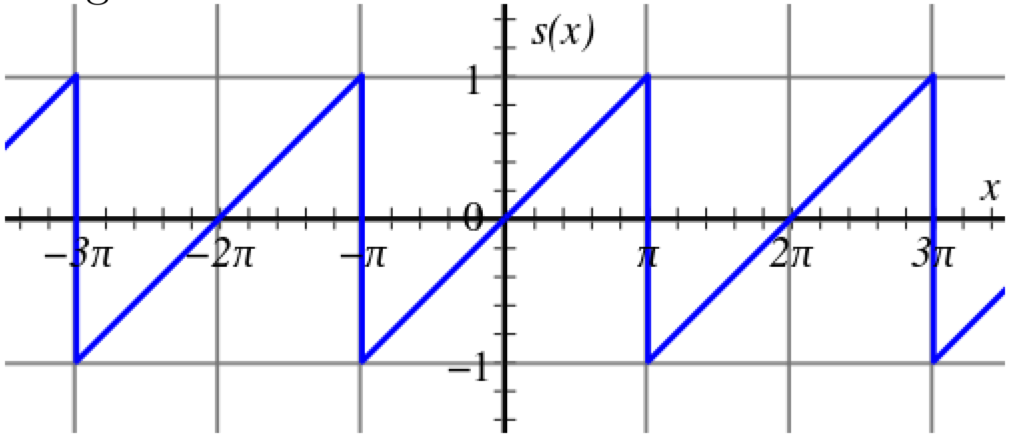
https://youtu.be/BSIw5SgUirg

FOURIER SERIES 1 SLIDE OVERVIEW

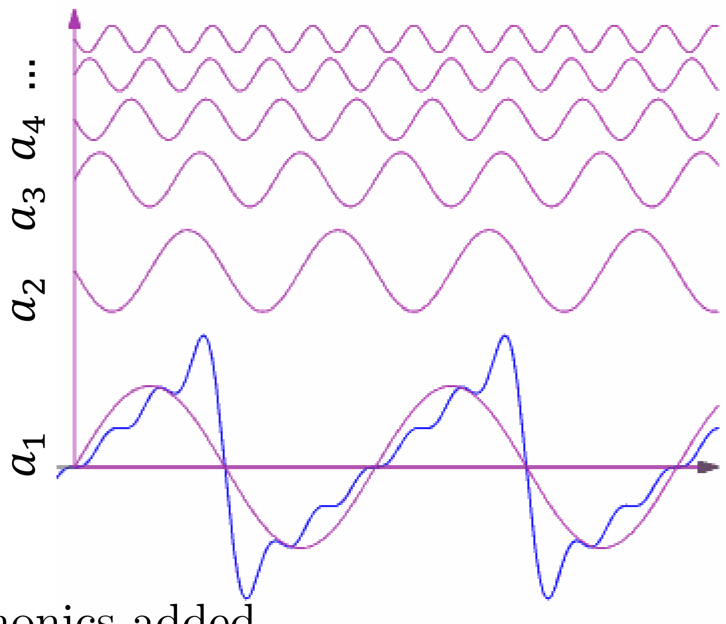
- Fourier argued that periodic signals (like the single period from a plucked string) were actually useful
  - Represent complex periodic signals
- Examples of basic periodic signals
  - Sinusoid: $x(t) = cos\omega_0 t$
  - Complex exponential: $x(t) = e^{j\omega_0 t}$
  - Fundamental frequency: ω_0
  - Fundamental period: $T = \frac{2\pi}{\omega_0}$
- Harmonically related period signals form family
  - Integer multiple of fundamental frequency
  - $\phi_k(t) = e^{jk\omega_0 t}$ for $k = 0, \pm 1, \pm 2, \ldots$
- Fourier Series is a way to represent a periodic signal as a linear combination of harmonics
  - $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
  - a_k coefficient gives the contribution of a harmonic (periodic signal of k times frequency)

SAWTOOTH EXAMPLE

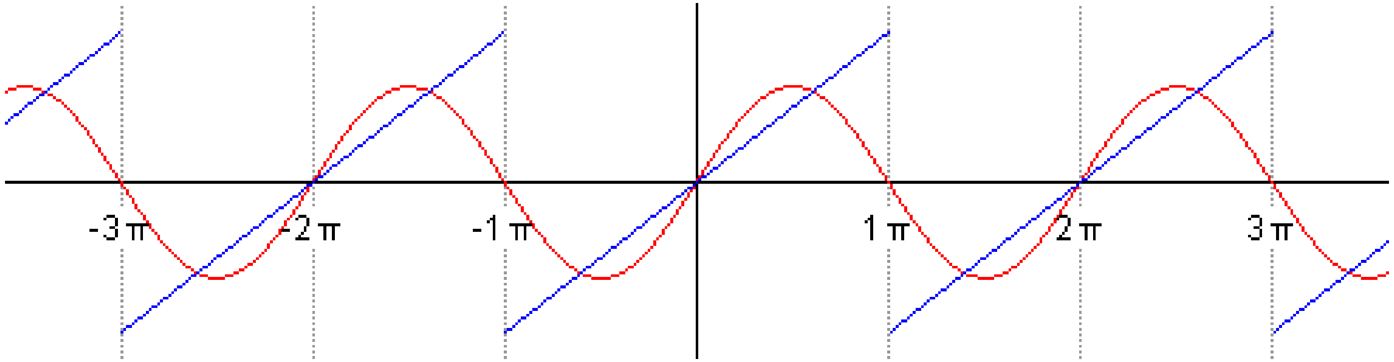
signal



Harmonics: height given by coefficient

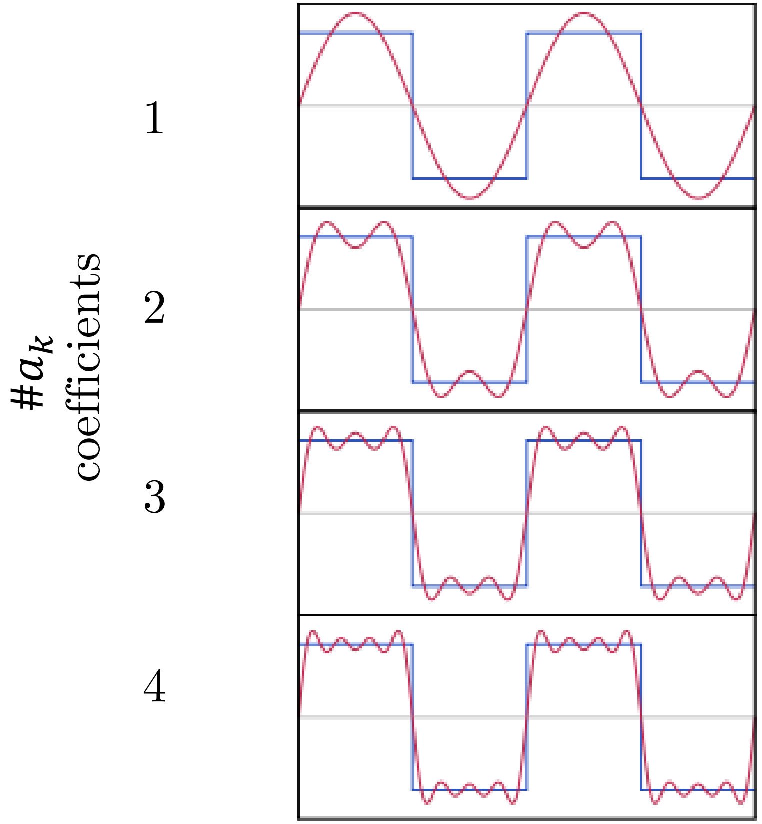


Animation showing approximation as more harmonics added

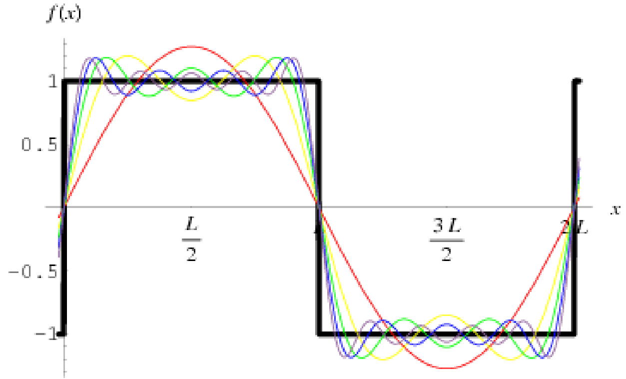


SQUARE WAVE EXAMPLE

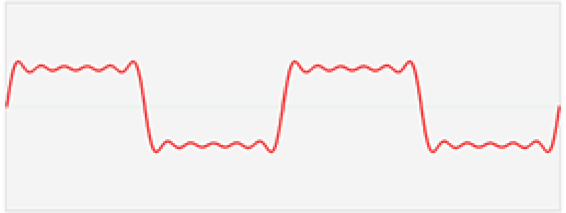
- Better approximation of square wave with more coefficients



- Aligned approximations

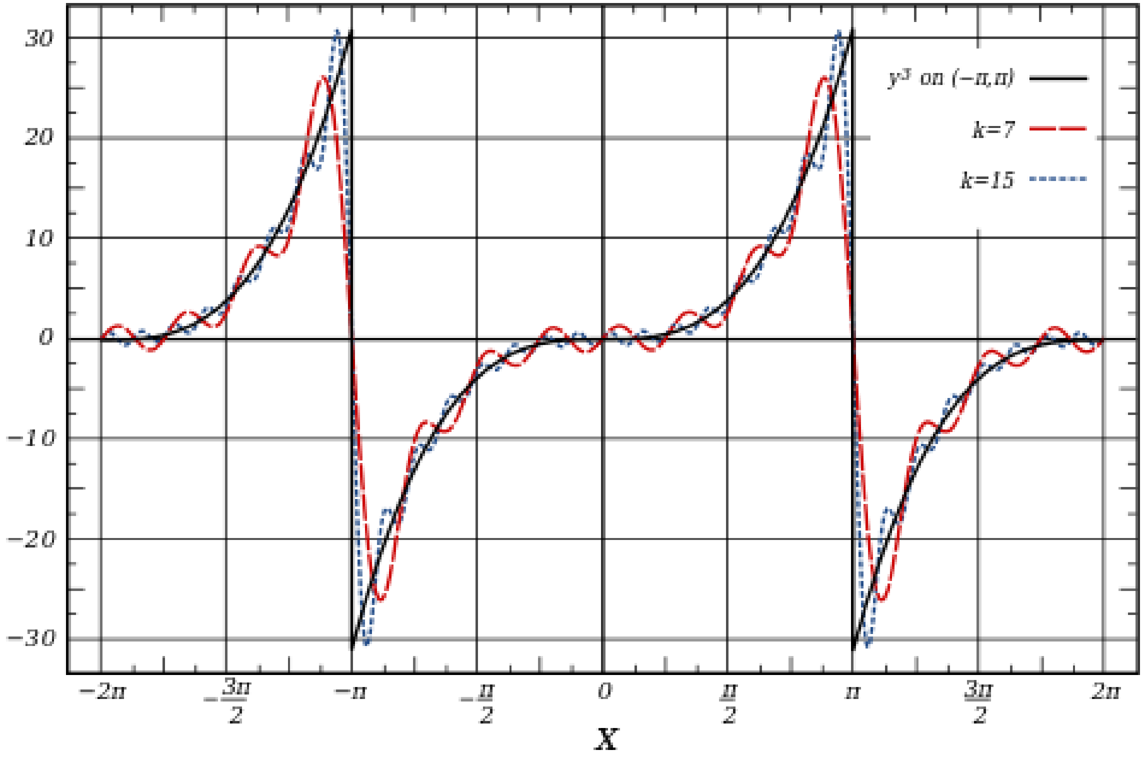


- Animation of FS



Note: $S(f) \sim a_k$

ARBITRARY EXAMPLES



- Interactive examples [flash (dated)][html]

RESPONSE OF LTI SYSTEMS TO COMPLEX EXPONENTIALS

CHAPTER 3.2

TRANSFORM ANALYSIS OBJECTIVE

- Need family of signals $\{x_k(t)\}$ that have 1) simple response and 2) represent a broad (useful) class of signals

1. Family of signals Simple response – every signal in family pass through LTI system with scale change

$$x_k(t) \longrightarrow \lambda_k x_k(t)$$

2. “Any” signal can be represented as a linear combination of signals in the family

$$x(t) = \sum_{k=-\infty}^{\infty} a_k x_k(t)$$

- Results in an output generated by input $x(t)$

$$x(t) \longrightarrow \sum_{k=-\infty}^{\infty} a_k \lambda_k x_k(t)$$

IMPULSE AS BASIC SIGNAL

- Previously (Ch2), we used shifted and scaled deltas
  - $\{\delta(t-t_0)\} \Longrightarrow x(t) = \int x(\tau)\delta(t-\tau)d\tau \longrightarrow y(t) = \int x(\tau)h(t-\tau)d\tau$

- Thanks to Jean Baptiste Joseph Fourier in the early 1800s we got Fourier analysis
  - Consider signal family of complex exponentials
    - $x(t) = e^{st}$ or $x[n] = z^n$, $s, z \in \mathbb{C}$

COMPLEX EXPONENTIAL AS EIGENSIGNAL

- Using the convolution
  - $e^{st} \longrightarrow H(s)e^{st}$
  - $z^n \longrightarrow H(z)z^n$

- Notice the eigenvalue $H(s)$ depends on the value of $h(t)$ and s
  - Transfer function of LTI system
  - Laplace transform of impulse response

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\
&= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}_{H(s)} \\
&= \underbrace{H(s)}_{\text{eigenvalue}} \cdot \underbrace{e^{st}}_{\text{eigenfunction}}
\end{aligned}
$$

TRANSFORM OBJECTIVE

- Simple response
  - $x(t) = e^{st} \longrightarrow y(t) = H(s)x(t)$
- Useful representation?
  - $x(t) = \sum a_k e^{s_k t} \longrightarrow y(t) = \sum a_k H(s_k) e^{s_k t}$
    - Input linear combination of complex exponentials leads to output linear combination of complex exponentials
  - Fourier suggested limiting to subclass of period complex exponentials $e^{jk\omega_0 t}, k \in \mathbb{Z}, \omega_0 \in \mathbb{R}$
  - $x(t) = \sum a_k e^{jk\omega_0 t} \longrightarrow y(t) = \sum a_k H(jk\omega_0) e^{s_k t}$
    - Periodic input leads to periodic output.
    - $H(j\omega) = H(s)|_{s=j\omega}$ is the frequency response of the system

CONTINUOUS TIME FOURIER SERIES

CHAPTER 3.3-3.8

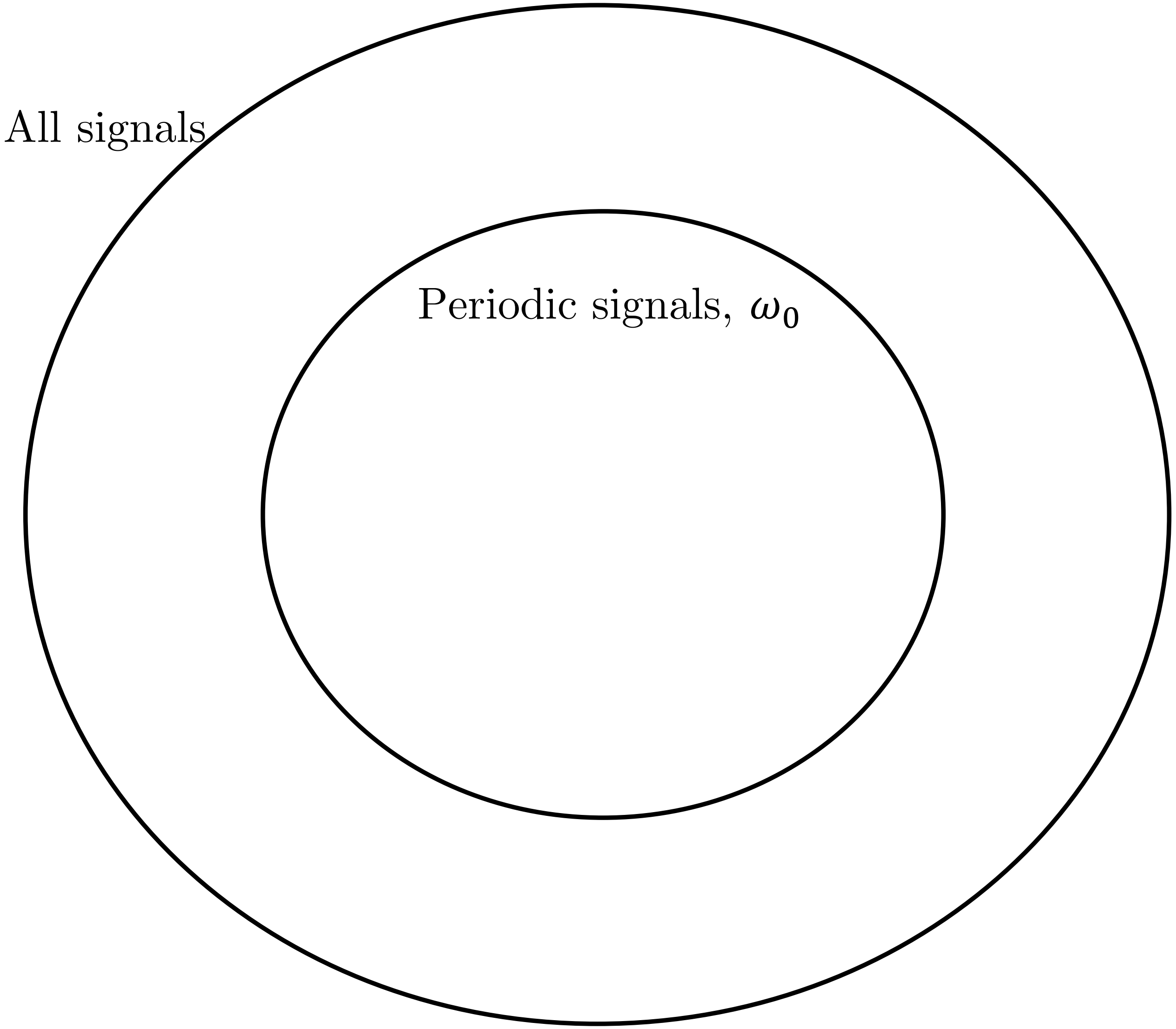
CTFS TRANSFORM PAIR

- Suppose $x(t)$ can be expressed as a linear combination of harmonic complex exponentials
  - $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ synthesis equation
- Then the FS coefficients $\{a_k\}$ can be found as
  - $a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt$ analysis equation
- ω_0 - fundamental frequency
- $T = 2\pi/\omega_0$ - fundamental period
- a_k known as FS coefficients or spectral coefficients

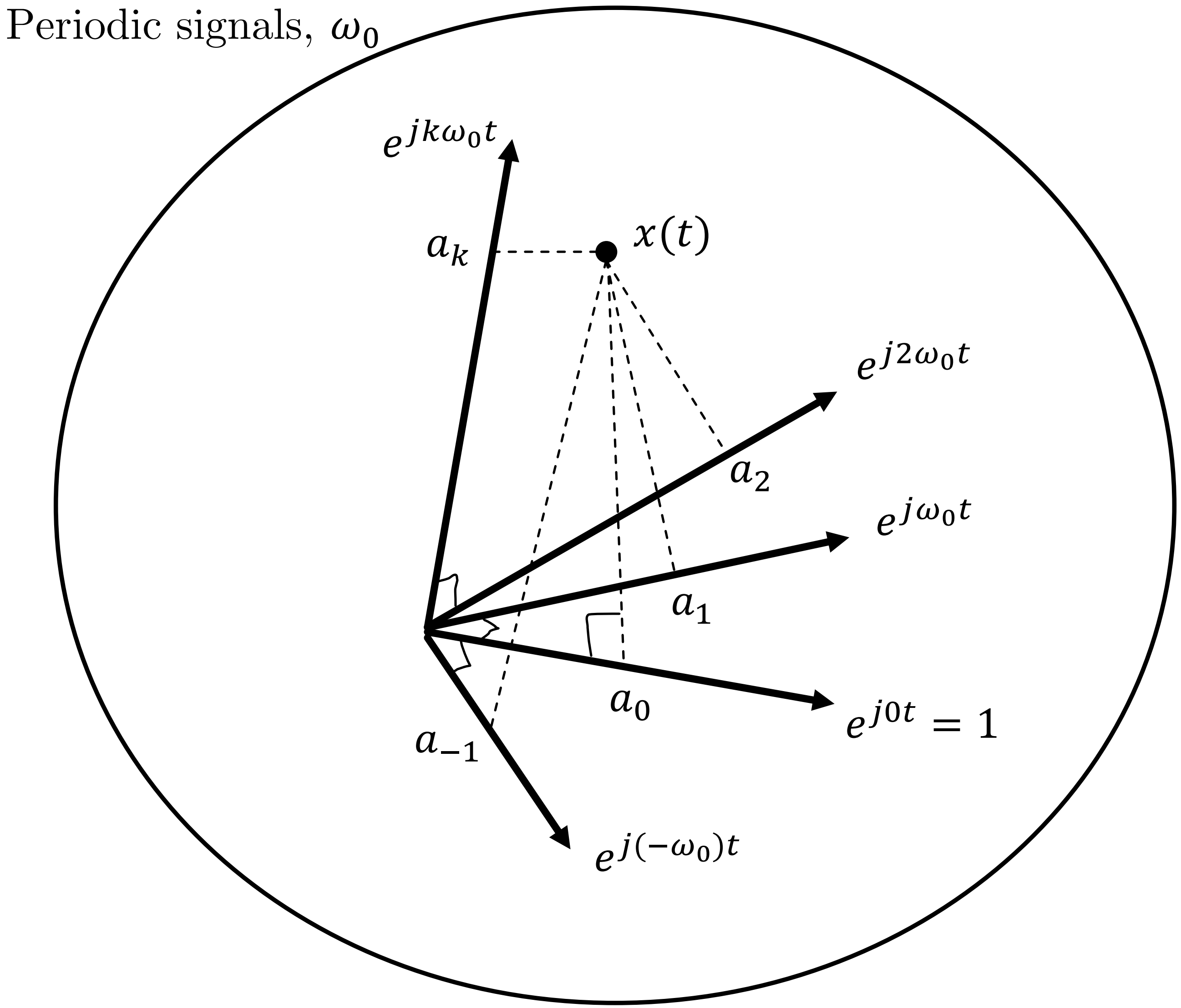
CTFS PROOF

- While we can prove this, it is not well suited for slides.
  - See additional handout for details
- Key observation from proof: Complex exponentials are orthogonal

VECTOR SPACE OF PERIODIC SIGNALS



VECTOR SPACE OF PERIODIC SIGNALS



- Each of the harmonic exponentials are orthogonal to each other and span the space of periodic signals
- The projection of $x(t)$ onto a particular harmonic (a_k) gives the contribution of that complex exponential to building $x(t)$
  - a_k is how much of each harmonic is required to construct the periodic signal $x(t)$

HARMONICS

- $k = \pm 1 \Rightarrow$ fundamental component (first harmonic)
  - Frequency ω_0, period $T = 2\pi/\omega_0$
- $k = \pm 2 \Rightarrow$ second harmonic
  - Frequency $\omega_2 = 2\omega_0$, period $T_2 = T/2$ (half period)
- …
- $k = \pm N \Rightarrow$ Nth harmonic
  - Frequency $\omega_N = N\omega_0$, period $T_N = T/N$ (1/N period)
- $k = 0 \Rightarrow a_0 = \frac{1}{T}\int_T x(t)dt$, DC, constant component, average over a single period

HOW TO FIND FS REPRESENTATION

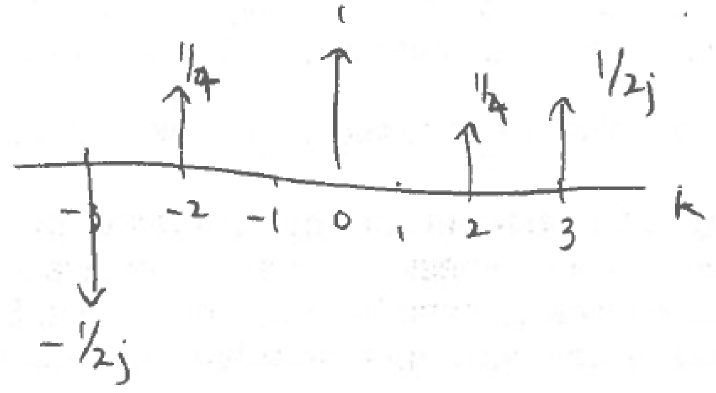
- Will use important examples to demonstrate common techniques

- Sinusoidal signals – Euler’s relationship
- Direct FS integral evaluation
- FS properties table and transform pairs

SINUSOIDAL SIGNAL

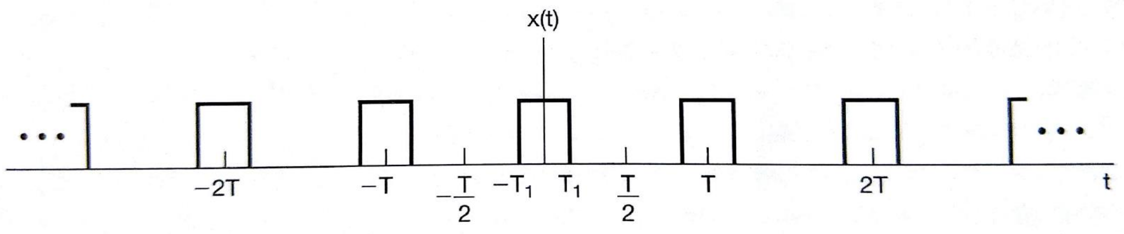
- $x(t) = 1 + \frac{1}{2}\cos 2\pi t + \sin 3\pi t$
- First find the period
  - Constant 1 has arbitrary period
  - $\cos 2\pi t$ has period $T_1 = 1$
  - $\sin 3\pi t$ has period $T_2 = 2/3$
  - $T = 2, \omega_0 = 2\pi/T = \pi$
- Rewrite $x(t)$ using Euler's and read off a_k coefficients by inspection

- $x(t) = 1 + \frac{1}{4}\left[e^{j2\omega_0 t} + e^{-j2\omega_0 t}\right] + \frac{1}{2j}\left[e^{j3\omega_0 t} - e^{-j3\omega_0 t}\right]$
- Read off coeff. directly
  - $a_0 = 1$
  - $a_1 = a_{-1} = 0$
  - $a_2 = a_{-2} = 1/4$
  - $a_3 = 1/2j$, $a_{-3} = -1/2j$
  - $a_k = 0$, else



PERIODIC RECTANGLE WAVE

- $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases}$



$k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{T}\int_T e^{-jk\omega_0 t}dt \\ &= -\frac{1}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1} = \frac{1}{jk\omega_0 T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right] \\ &= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \\ &= \underbrace{\frac{\sin(k\omega_0 T)}{k\pi}}_{\text{modulated sin function}} . \end{aligned}$$

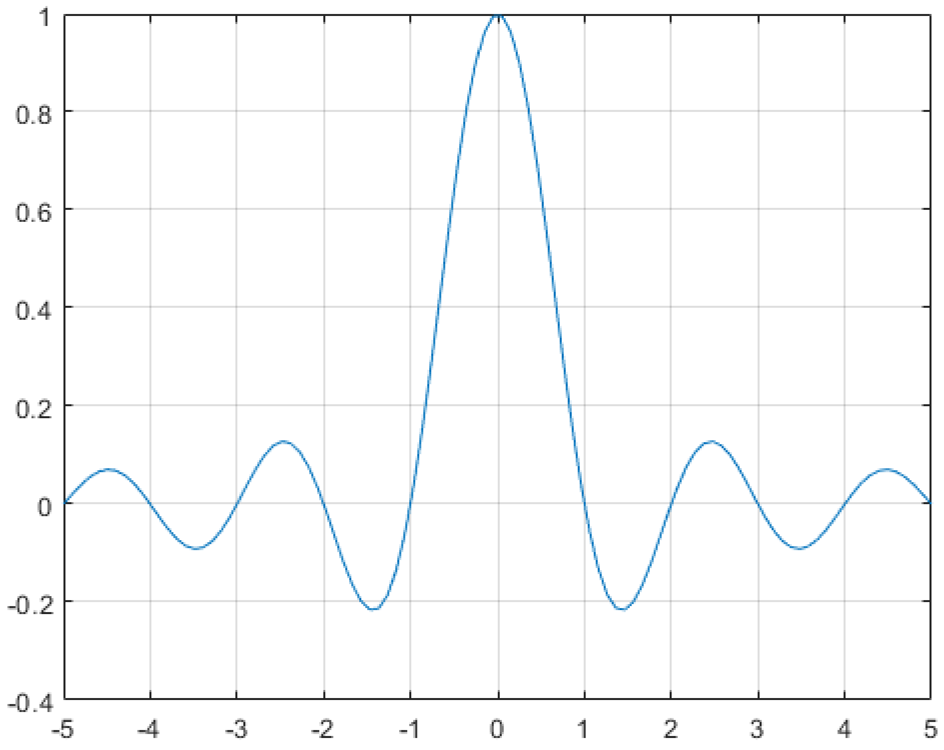
$k = 0$

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases} \longleftrightarrow a_k = \begin{cases} 2T_1/T & k = 0 \\ \frac{\sin(k\omega_0 T_1)}{k\pi} & k \neq 0 \end{cases}$$

SINC FUNCTION

- Important signal/function in DSP and communication
  - $\mathrm{sinc}(x) = \dfrac{\sin \pi x}{\pi x}$ normalized
  - $\mathrm{sinc}(x) = \dfrac{\sin x}{x}$ unnormalized
- Modulated sine function
  - Amplitude follows 1/x
  - Must use L'Hopital's rule to get x=0 time



RECTANGLE WAVE COEFFICIENTS

- Consider different "duty cycle" for the rectangle wave
  - $T = 4T_1$ 50% (square wave)
  - $T = 8T_1$ 25%
  - $T = 16T_1$ 12.5%
- Note all plots are still a sinc shape
  - Difference is how the sync is sampled
  - Longer in time (larger T) smaller spacing in frequency → more samples between zero crossings

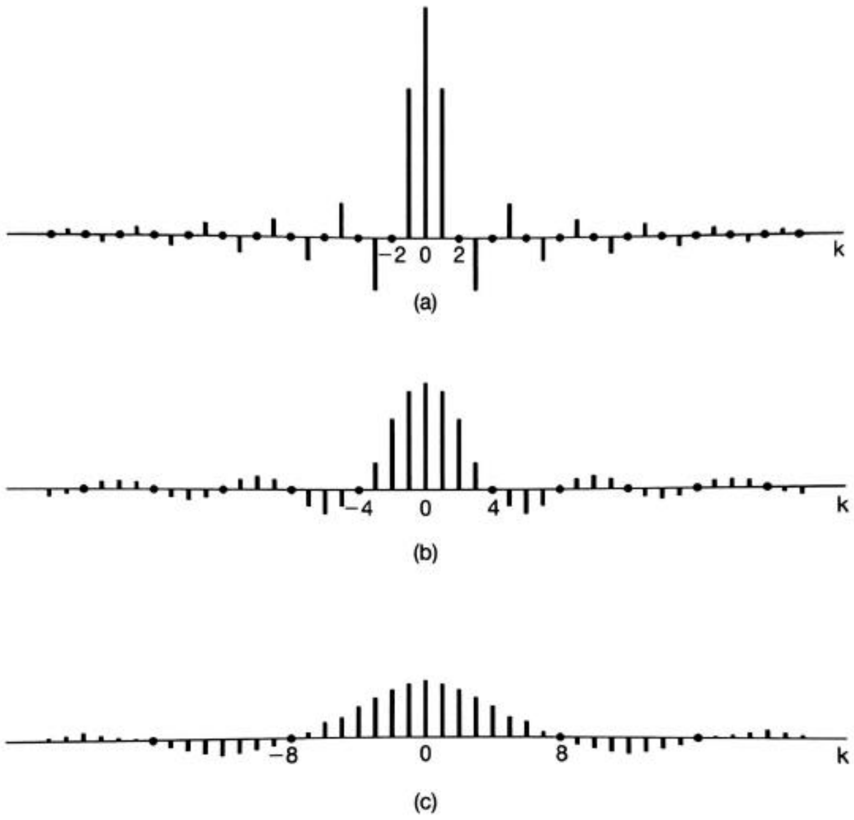
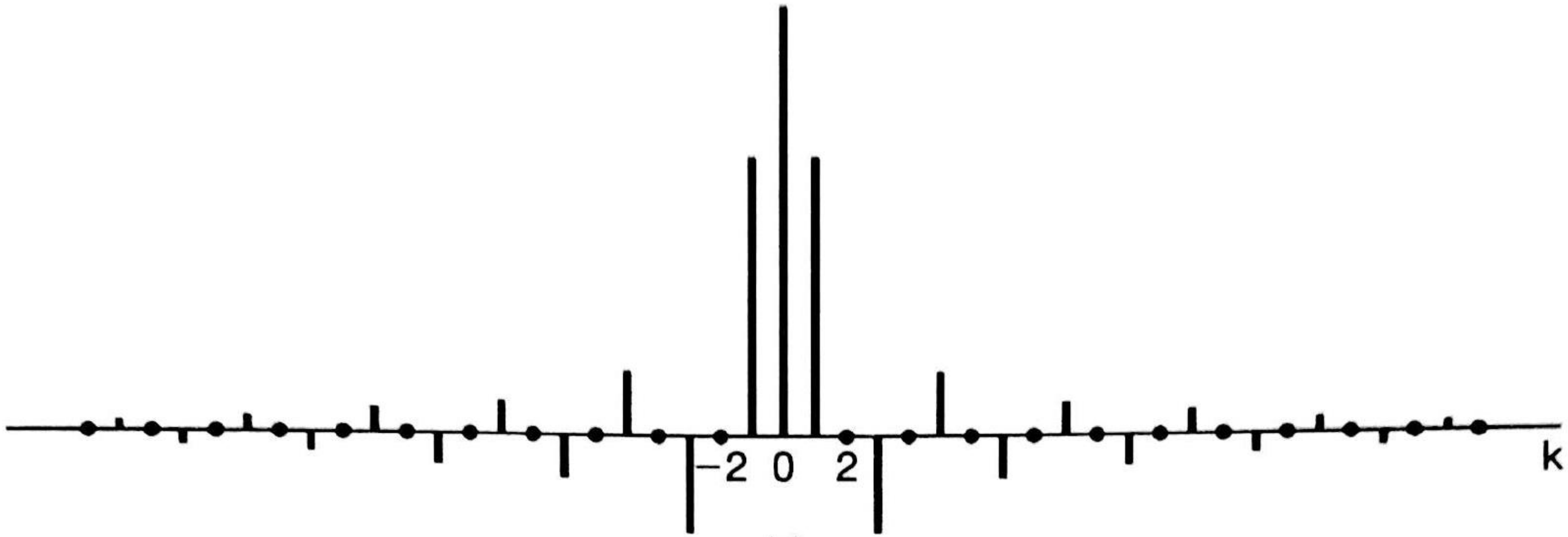


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

SQUARE WAVE

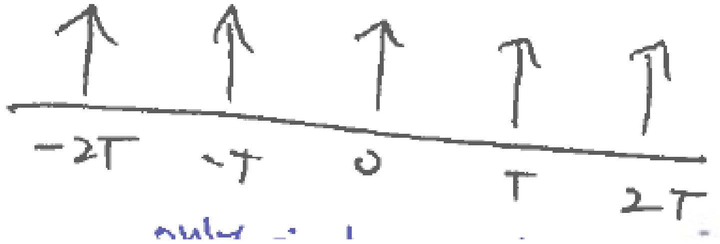
- Special case of rectangle wave with $T = 4T_1$
  - One sample between zero-crossing
- $a_k = \begin{cases} 1/2 & k = 0 \\ \dfrac{\sin(k\pi/2)}{k\pi} & else \end{cases}$



(a)

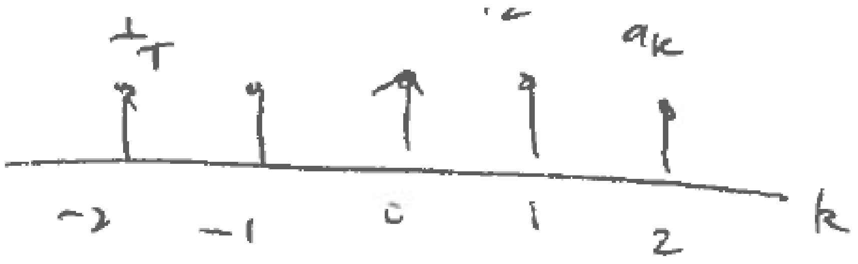
PERIODIC IMPULSE TRAIN

- $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$
- Using FS integral



$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$
$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum \delta(t - kT)e^{-jk\omega_0 t}dt$$

- Notice only one impulse in the interval



$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt$$
$$a_k = \frac{1}{T}\underbrace{\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 0}\,dt}_{=1} = \frac{1}{T}$$

PROPERTIES OF CTFS

- Since these are very similar between CT and DT, will save until after DT

- Note: As for LT and Z Transform, properties are used to avoid direct evaluation of FS integral
  - Be sure to bookmark properties in Table 3.1 on page 206

DISCRETE TIME FOURIER SERIES

CHAPTER 3.6

DTFS VS CTFS DIFFERENCES

- While quite similar to the CT case,
  - DTFS is a finite series, $\{a_k\}, |k| < K$
  - Does not have convergence issues
- Good News: motivation and intuition from CT applies for DT case

DTFS TRANSFORM PAIR

- Consider the discrete time periodic signal $x[n] = x[n+N]$

- $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$ synthesis equation
- $a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$ analysis equation

- N – fundamental period (smallest value such that periodicity constraint holds)
- $\omega_0 = 2\pi/N$ – fundamental frequency
- $\sum_{n=<N>}$ indicates summation over a period (N samples)

DTFS REMARKS

- DTFS representation is a finite sum, so there is always pointwise convergence
- FS coefficients are periodic with period N

DTFS PROOF

- Proof for the DTFS pair is similar to the CT case
- Relies on orthogonality of harmonically related DT period complex exponentials
- Will not show in class

HOW TO FIND DTFS REPRESENTATION

- Like CTFS, will use important examples to demonstrate common techniques

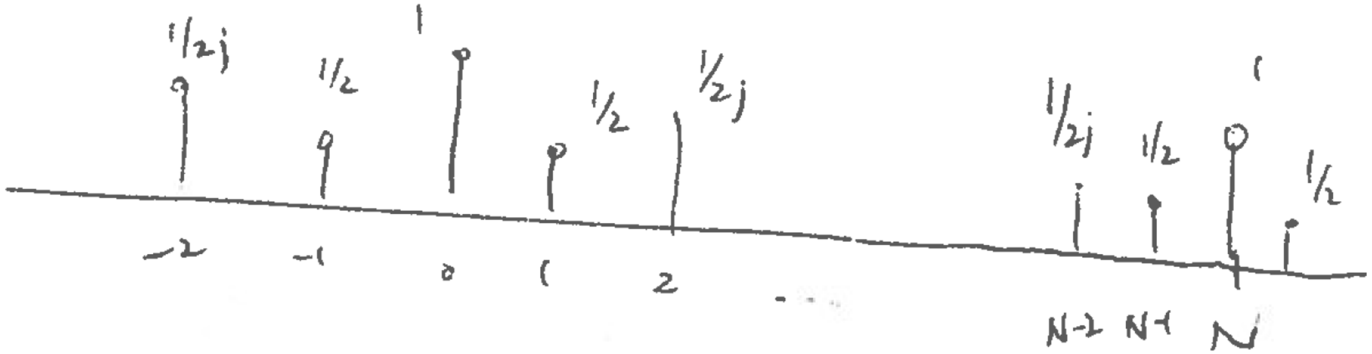
- Sinusoidal signals – Euler's relationship
- Direct FS summation evaluation – periodic rectangular wave and impulse train
- FS properties table and transform pairs

SINUSOIDAL SIGNAL

- $x[n] = 1 + \frac{1}{2}\cos\left(\frac{2\pi}{N}\right)n + \sin\left(\frac{4\pi}{N}\right)n$

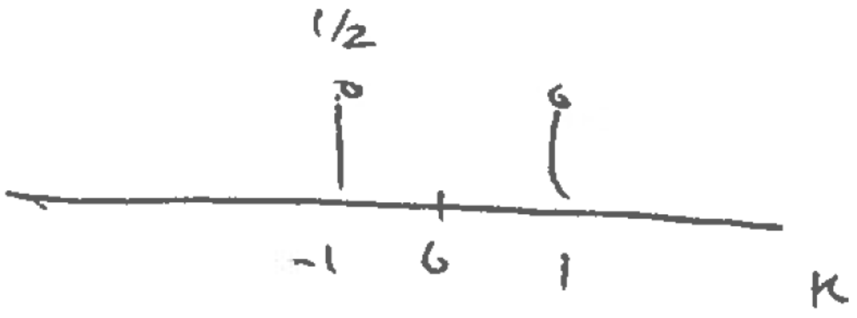
$$
\begin{aligned}
x[n] &= 1 + \frac{1}{2}\cos\left(\frac{2\pi}{N}\right)n + \sin\left(\frac{4\pi}{N}\right)n \\
&= 1 + \frac{1}{4}\left(e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right) + \frac{1}{2j}\left(e^{j\frac{4\pi}{N}n} - e^{-j\frac{4\pi}{N}n}\right) \\
&= 1 + \frac{1}{4}\left(e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right) + \frac{1}{2j}\left(e^{j2\frac{2\pi}{N}n} - e^{-j2\frac{2\pi}{N}n}\right)
\end{aligned}
$$

- First find the period
- Rewrite $x[n]$ using Euler's and read off a_k coefficients by inspection
- $a_0 = 1, a_{\pm 1} = \frac{1}{4}, a_2 = a_{-2}^* = \frac{1}{2j}$
- Shortcut here

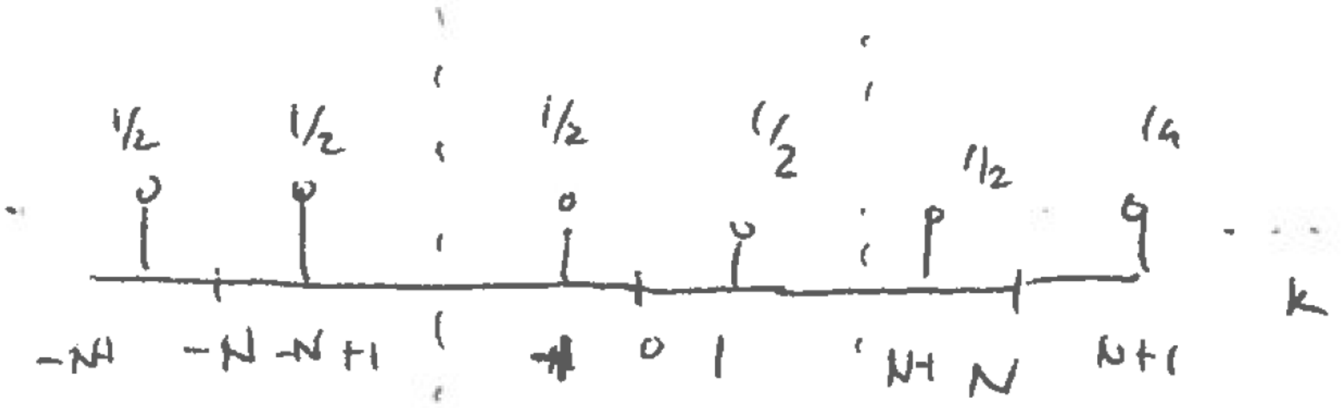


SINUSOIDAL COMPARISON

- $x(t) = \cos\omega_0 t$
- $a_k = \begin{cases} 1/2 & k = \pm 1 \\ 0 & else \end{cases}$



- $x[n] = \cos\omega_0 n$
- $a_k = \begin{cases} 1/2 & k = \pm 1 \\ 0 & else \end{cases}$
  - Over a single period → must specify period with period N



PERIODIC RECTANGLE WAVE

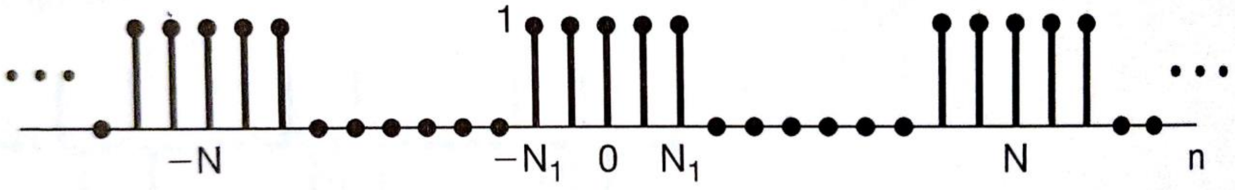


Figure 3.16 Discrete-time periodic square wave.

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=-N/2}^{N/2-1} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} \alpha^n$$

Remember the truncated geometric series $\sum_{n=0}^{N-1}\alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$$k = \begin{matrix} 0 \\ \pm N \\ \pm 2N \\ \vdots \end{matrix} \qquad a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1 = \frac{2N_1+1}{N}$$

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1}\alpha^{m-N_1}$$

$$= \frac{1}{N}\alpha^{-N_1}\sum_{m=0}^{2N_1}\alpha^m = \frac{1}{N}\alpha^{-N_1}\left(\frac{1-\alpha^{2N_1+1}}{1-\alpha}\right)$$

$$= \frac{1}{N}e^{-jk\omega_0 N_1}\left(\frac{1-e^{jk\omega_0(2N_1+1)}}{1-e^{-jk\omega_0}}\right)$$

$$= \dots$$

$$= \frac{\sin 2\pi k\left(N_1+\frac{1}{2}\right)/N}{\sin k\omega_0/2} = \frac{\sin 2\pi k(N_1+1/2)/N}{\sin k\pi/N}$$

$$x[n] = \begin{cases} 1 & |n| < N_1 \\ 0 & N_1 < |n| < N/2 \end{cases}$$

$$\updownarrow$$

$$a_k = \begin{cases} (2N_1+1)/N & k = 0, \pm N, \pm 2N, \dots \\ \frac{\sin 2\pi k(N_1+1/2)/N}{\sin k\pi/N} & k \neq 0, \pm N, \pm 2N, \dots \end{cases}$$

RECTANGLE WAVE COEFFICIENTS

- Consider different "duty cycle" for the rectangle wave
  - 50% (square wave)
  - 25%
  - 12.5%
- Note all plots are still a sinc shaped, but periodic
  - Difference is how the sync is sampled
  - Longer in time (larger N) smaller spacing in frequency → more samples between zero crossings

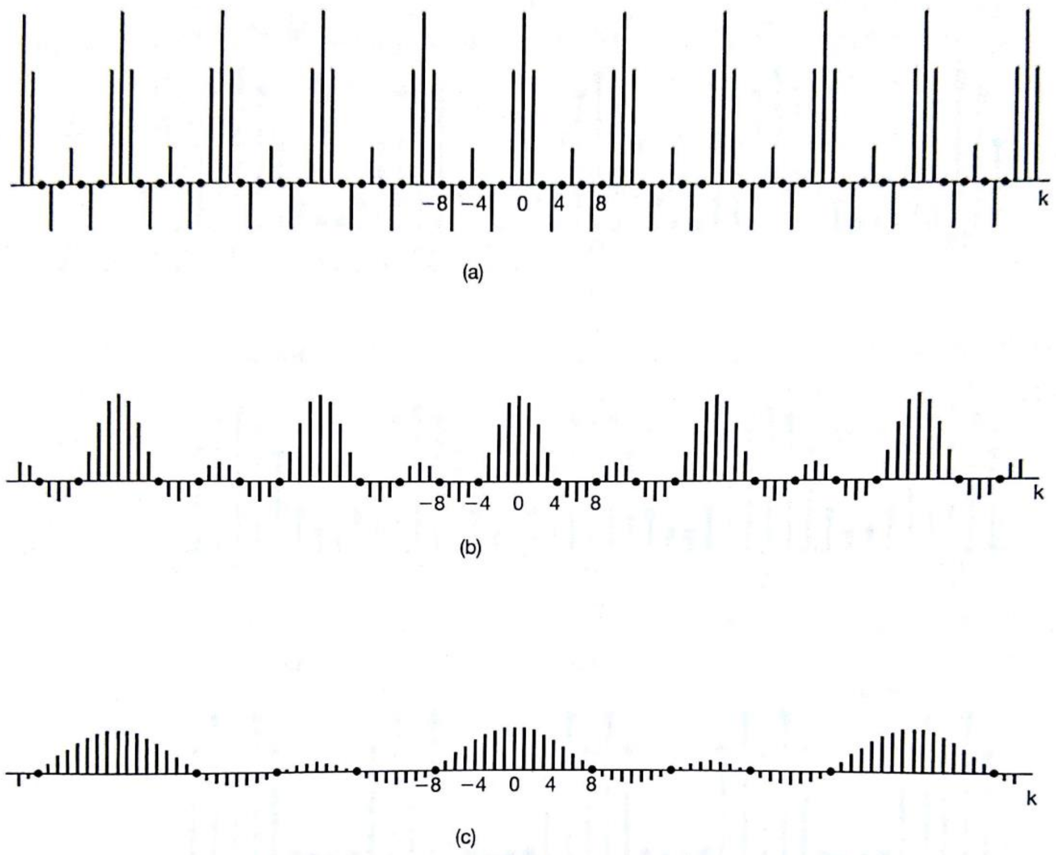
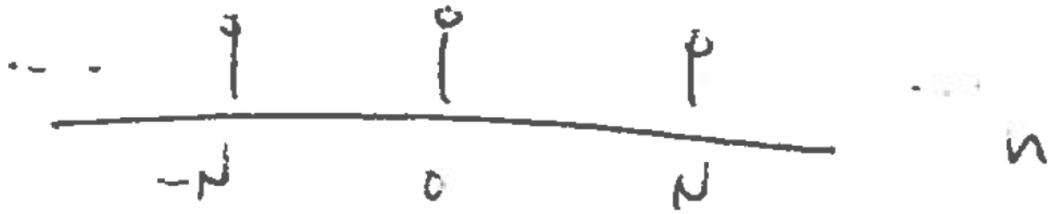


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

PERIODIC IMPULSE TRAIN

- $x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$
- Using FS integral



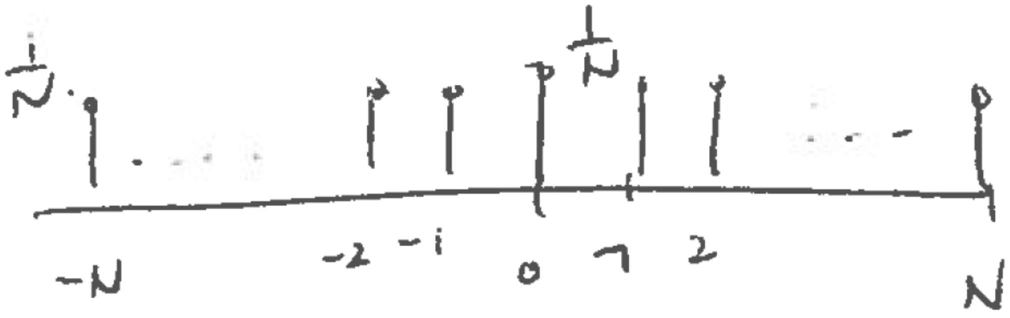
$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} dt$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum \delta[n - kN] e^{-jk\omega_0 n} dt$$

- Notice only one impulse in the interval

$$= \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-jk\omega_0 n} dt$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-jk\omega_0 0} dt = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] = \frac{1}{N}$$



PROPERTIES OF FOURIER SERIES

CHAPTER 3.5, 3.7

PROPERTIES OF FOURIER SERIES

- See Table 3.1 pg. 206 (CT) and Table 3.2 pg. 221 (DT)

- In the following slides, suppose:

$$x(t) \overset{\text{FS}}{\longleftrightarrow} a_k \qquad\qquad x[n] \overset{\text{FS}}{\longleftrightarrow} a_k$$

$$y(t) \overset{\text{FS}}{\longleftrightarrow} b_k \qquad\qquad y[n] \overset{\text{FS}}{\longleftrightarrow} b_k$$

- Most times, will only show proof for one of CT or DT

LINEARITY

- CT
- $Ax(t) + By(t) \longleftrightarrow Aa_k + Bb_k$
- DT
- $Ax[n] + By[n] \longleftrightarrow Aa_k + Bb_k$

TIME-SHIFT

- CT
- $x(t-t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$
- DT
- $x[n-n_0] \leftrightarrow a_k e^{-jk\omega_0 n_0}$
- Proof
  - Let $y(t) = x(t-t_0)$

$$b_k = \frac{1}{T}\int_T y(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt$$

Let $\tau = t - t_0$

$$= \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau$$

$$= e^{-jk\omega_0 t_0}\underbrace{\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau}_{a_k} = e^{-jk\omega_0 t_0}a_k$$

FREQUENCY SHIFT

- CT
- $e^{jM\omega_0 t}x(t) \leftrightarrow a_{k-M}$

- DT
- $e^{jM\omega_0 n}x[n] \leftrightarrow a_{k-M}$

Note: Similar relationship with Time Shift (dualilty). Multiplication by exponential in time is a shift in frequency. Shift in time is a multiplication by exponential in frequency.

TIME REVERSAL

- CT
- $x(-t) \leftrightarrow a_{-k}$

Proof, let $y(t) = x(-t)$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = x(-t)$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 -t}$$

Let $m = -k$

$$= \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega_0 t}$$

$$\Rightarrow b_k = a_{-k}$$

- DT
- $x[-n] \leftrightarrow a_{-k}$

PERIODIC CONVOLUTION

- CT
- $\int_T x(\tau)y(t-\tau)d\tau \leftrightarrow Ta_k b_k$
- DT
- $\sum_{r=<N>} x[r]y[n-r] \leftrightarrow Na_k b_k$

MULTIPLICATION

- CT
- $x(t)y(t) \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k$

- DT
- $x[n]y[n] \leftrightarrow \sum_{l=<N>} a_l b_{k-l} = a_k * b_k$
  - Convolution over a single period (DT FS is periodic)

Note: Similar relationship with Convolution (dualilty). Convolution in time results in multiplication in frequency domain. Multiplication in time results in convolution in frequency domain.

PARSEVAL'S RELATION

- CT
- $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

- DT
- $\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$

Note: Total average power in a periodic signal equals the sum of the average power in all its harmonic components

$$\frac{1}{\mathrm{T}}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2$$

Average power in the kth harmonic

TIME SCALING

- CT
- $x(\alpha t) \longleftrightarrow a_k$
  - $\alpha > 0$
  - Periodic with period T/α

- DT
- $x_{(m)}[n] = \begin{cases} x[n/m] & n \text{ multiple of } m \\ 0 & else \end{cases}$
  - Periodic with period mN
- $x_{(m)}[n] \longleftrightarrow \frac{1}{m} a_k$
  - Periodic with period mN

Note: Not all properties are exactly the same. Must be careful due to constraints on periodicity for DT signal.

FOURIER SERIES AND LTI SYSTEMS

CHAPTER 3.8

EIGENSIGNAL REMINDER

- $x(t) = e^{st} \leftrightarrow y(t) = H(s)e^{st}$ $\quad$ $x[n] = z^n \leftrightarrow y[n] = H(z)z^n$
- $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$ $\quad$ $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-k}$
- $H(s), H(z)$ known as system function $(s, z \in \mathbb{C})$

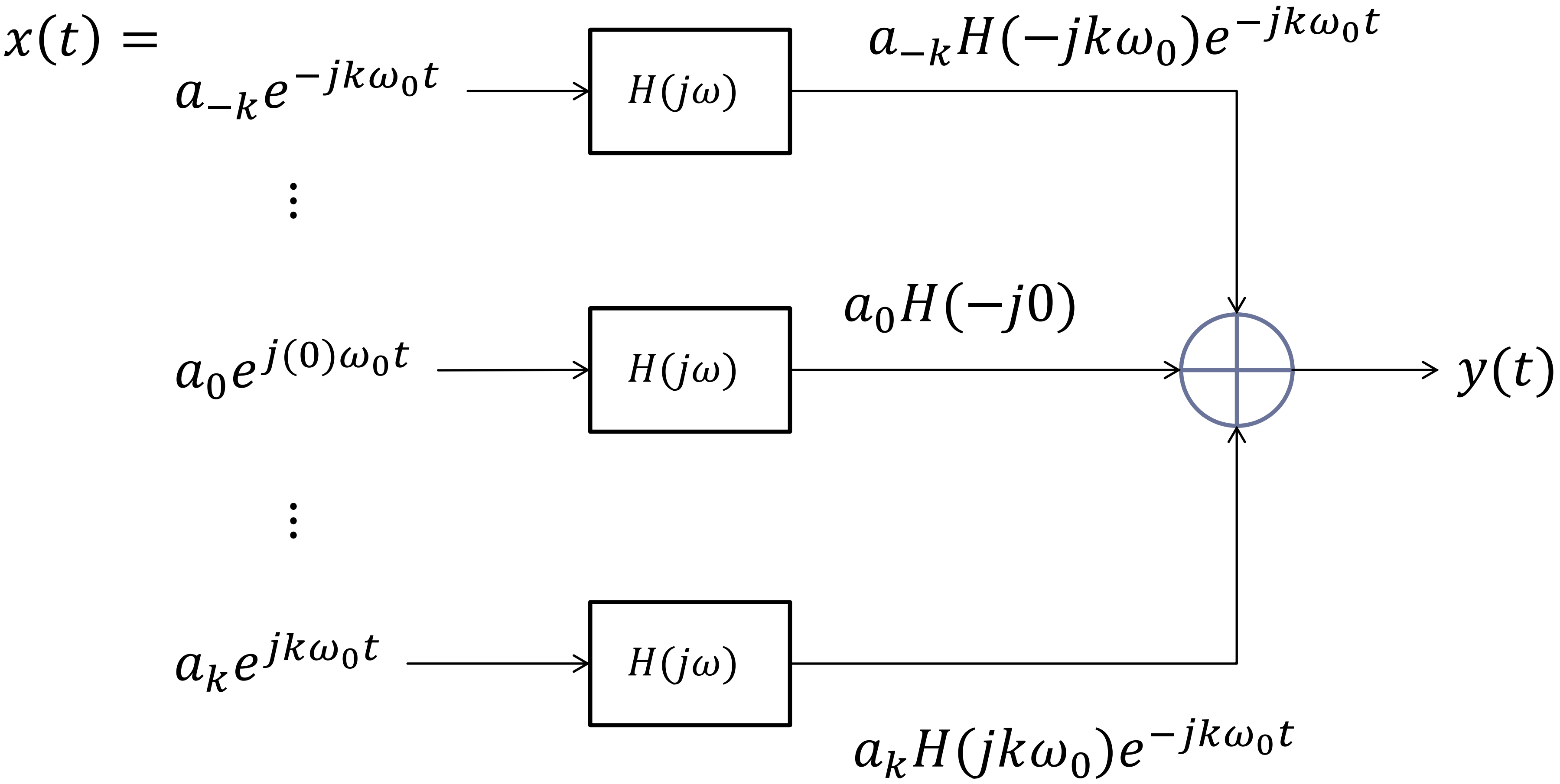
- For Fourier Analysis (e.g. FS)
  - Let $s = j\omega$ and $z = e^{j\omega}$
- Frequency response (system response to particular input frequency)
  - $H(j\omega) = H(s)|_{s=j\omega} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$
  - $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$

FOURIER SERIES AND LTI SYSTEMS I

- Consider now a FS representation of a periodic signals
- $x(t) = \sum_k a_k e^{jk\omega_0 t}$
- $\rightarrow y(t) = \sum_k a_k H(jk\omega_0) e^{jk\omega_0 t}$
  - Due to superposition (LTI system)
  - Each harmonic in results in harmonic out with eigenvalue
- $y(t)$ periodic with same fundamental frequency as $x(t) \Rightarrow \omega_0$
  - $T = \frac{2\pi}{\omega_0}$ - fundamental period
- FS coefficients for $y(t)$
  - $b_k = a_k H(jk\omega_0)$
  - b_k is the FS coefficient a_k multiplied/affected by frequency response at $k\omega_0$

FOURIER SERIES AND LTI SYSTEMS III

- System block diagram



DTFS AND LTI SYSTEMS

- $x[n] = \sum_{k=<N>} a_k e^{jk2\pi/Nn} \rightarrow$

$$y[n] = \sum_{k=<N>} a_k H(e^{j\frac{2\pi}{N}k}) e^{jk2\pi/Nn}$$

- Same idea as in the continuous case
  - Each harmonic is modified by the Frequency Response at the harmonic frequency

EXAMPLE 1

- LTI system with
  - $h[n] = \alpha^n u[n], -1 < \alpha < 1$
- Find FS of $y[n]$ given input
  - $x[n] = \cos\frac{2\pi n}{N}$
- Find FS representation of $x[n]$
  - $\omega_0 = 2\pi/N$
  - $x[n] = \frac{1}{2}e^{j2\pi/Nn} + \frac{1}{2}e^{-j2\pi/Nn}$
  - $a_k = \begin{cases} \frac{1}{2} & k = \pm 1, \pm(N+1), \dots \\ 0 & \text{else} \end{cases}$

- Find frequency response
  - $H(e^{j\omega}) = \sum_n h[n]e^{-j\omega n}$
  - $H(e^{j\omega}) = \sum_n \alpha^n u[n]e^{-j\omega n}$

$$H(j\omega) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$H(j\omega) = \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n$$

Let $\beta = \alpha e^{-j\omega}$

$$H(j\omega) = \frac{1}{1-\beta}$$

$$H(j\omega) = \frac{1}{1-\alpha e^{-j\omega}}$$

EXAMPLE 1 II

- Use FS LTI relationship to find output
  - $y[n] = \sum_{k=<N>} a_k H(e^{jk\omega_0})\, e^{jk\omega_0 n}$
  - $y[n] = \frac{1}{2} H\left(e^{j1\frac{2\pi}{N}n}\right) e^{j1\frac{2\pi}{N}n} + \frac{1}{2} H\left(e^{-j1\frac{2\pi}{N}n}\right) e^{-j1\frac{2\pi}{N}n}$
  - $y[n] = \frac{1}{2}\left(\frac{1}{1-\alpha e^{-jk2\pi/N}}\right) e^{j\frac{2\pi}{N}n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{jk2\pi/N}}\right) e^{-j\frac{2\pi}{N}n}$
- Output FS coefficients
  - $b_k = \begin{cases} \frac{1}{2}\left(\frac{1}{1-\alpha e^{-jk2\pi/N}}\right) & k = \pm 1 \\ 0 & else \end{cases}$

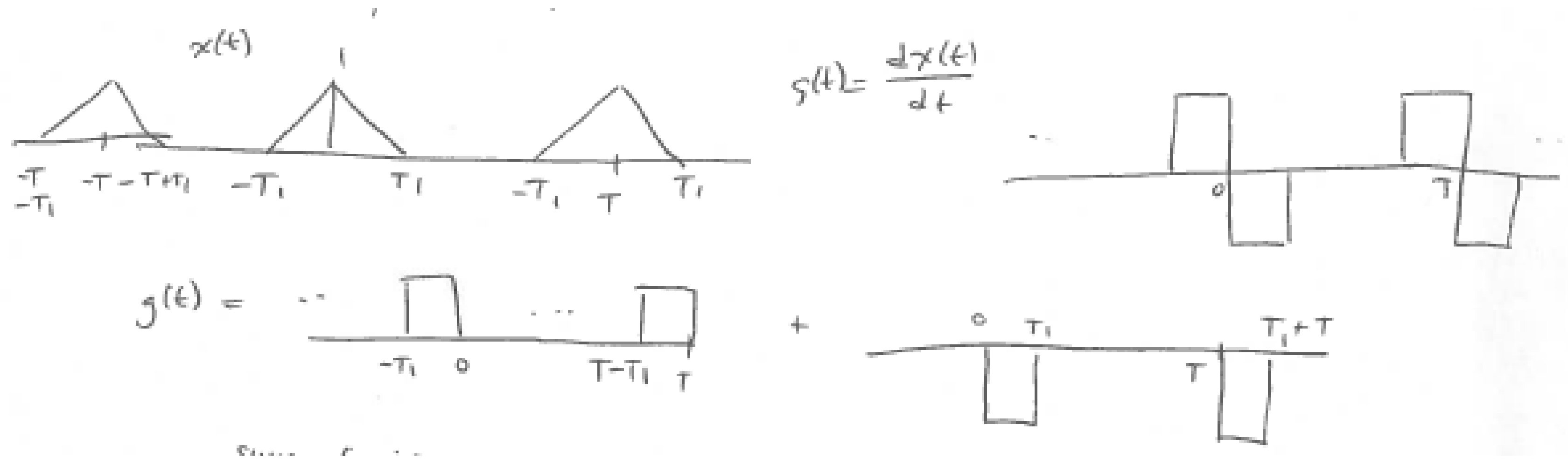
Periodic with period N

EXAMPLE PROBLEM 3.7

- $x(t)$ has fundamental period T and FS a_k
- Sometimes direct calculation of a_k is difficult, at times easier to calculate transformation
  - $b_k \leftrightarrow g(t) = \frac{dx(t)}{dt}$
- Find a_k in terms of b_k and T, given
  - $\int_T^{2T} x(t)dt = 2$
- $a_0 = \frac{1}{T}\int_T x(t)e^{-j(0)\omega_0 t}dt = \frac{1}{T}\int_T x(t)dt \Rightarrow \frac{2}{T}$
- From Table 3.1 pg 206
  - $b_k \leftrightarrow jk\frac{2\pi}{T}a_k \Rightarrow a_k = \frac{b_k}{jk2\pi/T}$
- $a_k = \begin{cases} 2/T & k = 0 \\ \frac{b_k}{jk2\pi/T} & k \neq 0 \end{cases}$

EXAMPLE PROBLEM 3.7 II

- Find FS of periodic sawtooth wave



- Take derivative of sawtooth
  - Results in sum of rectangular waves
- FS coefficients of rectangular waves from Table 3.2 to get $b_k \leftrightarrow g(t)$
- Then use previous result to find $a_k \leftrightarrow x(t)$
- See examples 3.6, 3.7 for similar treatment

CHAPTER 3.9

FILTERING

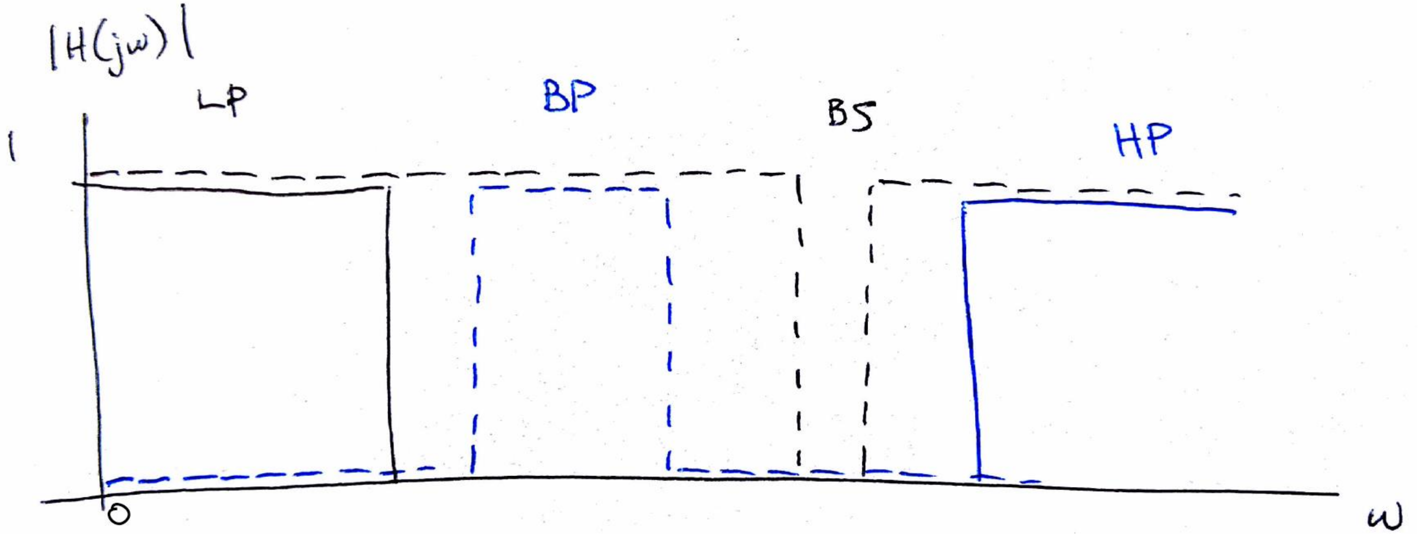
FILTERING

- Important process in many applications

- The goal is to change the relative amplitudes of frequency components in a signal
  - In EE480: DSP you can learn how to design a filter with desired properties/specifications

LTI FILTERS

- Frequency-shaping filters – general LTI systems
- Frequency-selective filters – pass some frequencies and eliminate others
  - Common examples include low-pass (LP), high-pass (HP), bandpass (BP), and bandstop (BS) [notch]



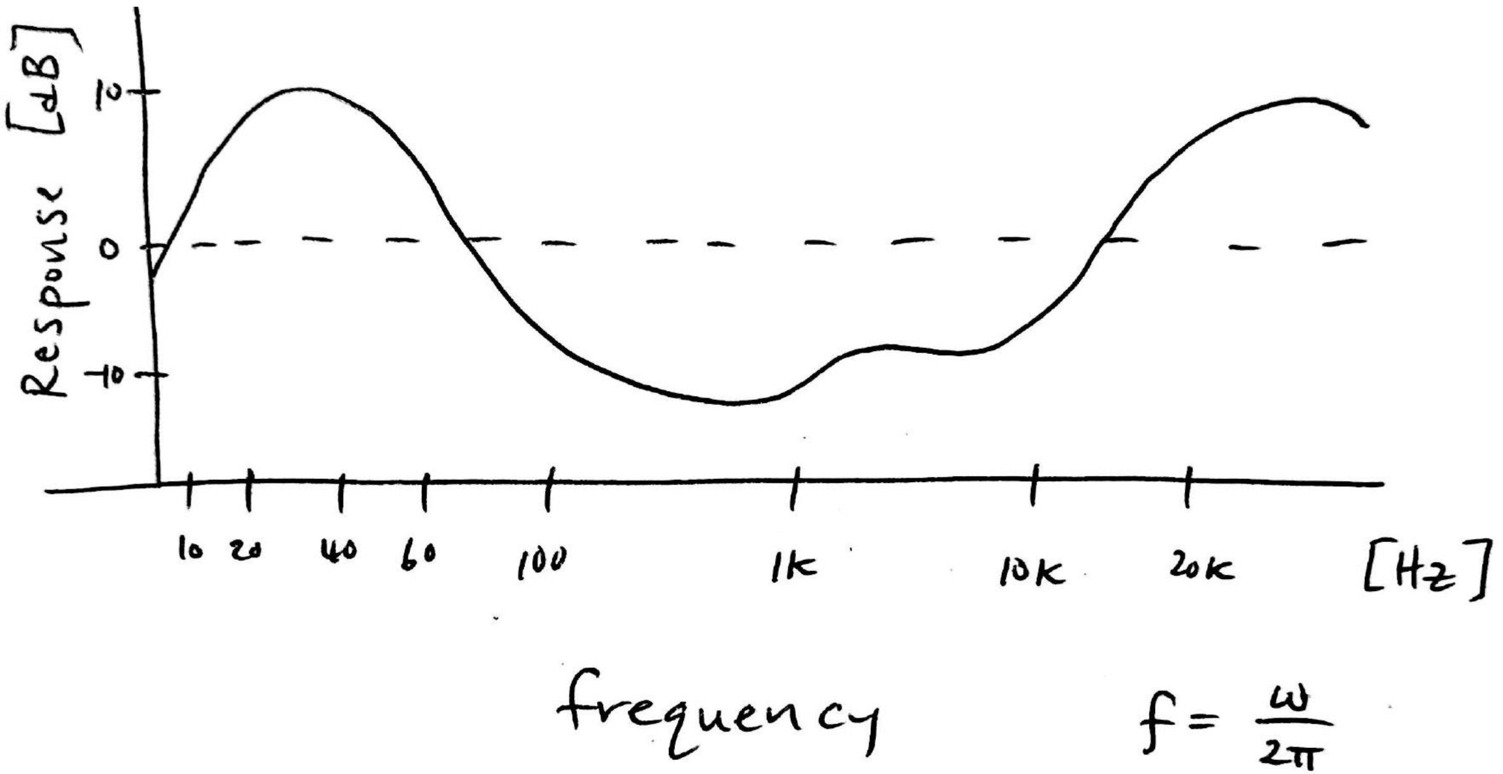
MOTIVATION: AUDIO EQUALIZER

- Basic equalizer gives user ability to adjust sound from to match taste – e.g. bass (low freq) and treble (high freq)

  - Log-log plot to show larger range of frequencies and response

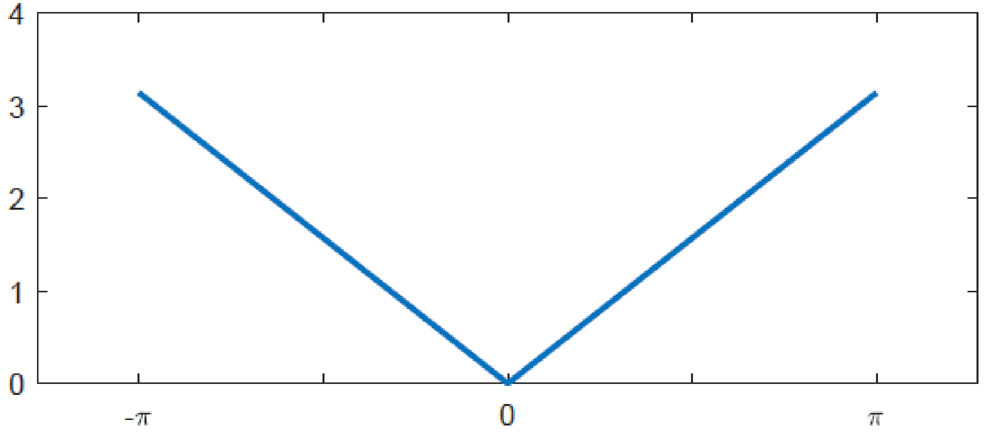
    $$\mathrm{dB} = 20\log_{10}|H(j\omega)|$$

  - Magnitude response matches are intuition
    - Boost low and high frequencies but attenuate mid frequencies

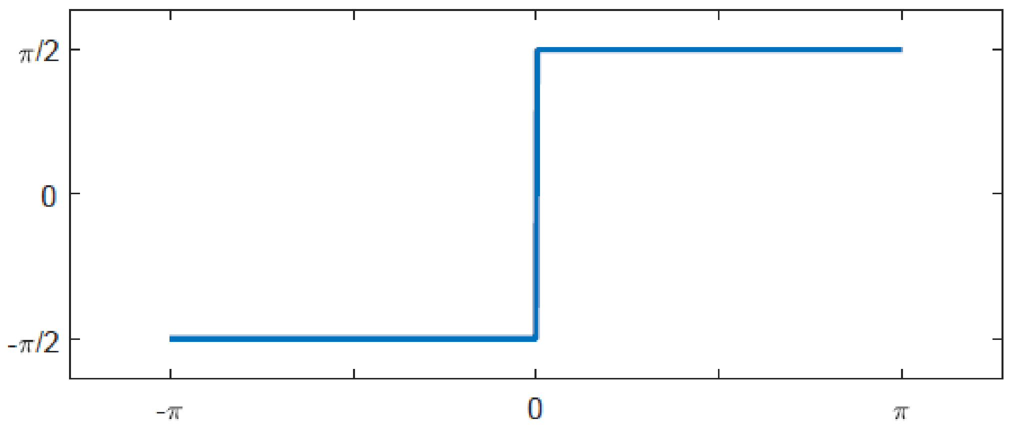


EXAMPLE: DERIVATIVE FILTER

- $y(t) = \frac{d}{dt}x(t) \quad \leftrightarrow \quad H(j\omega) = j\omega$
- High-pass filter used for “edge” detection



(a) $|H(j\omega)| = |\omega|$



(b) $\angle H(j\omega) = \tan^{-1}\left(\frac{Im}{Re}\right)$

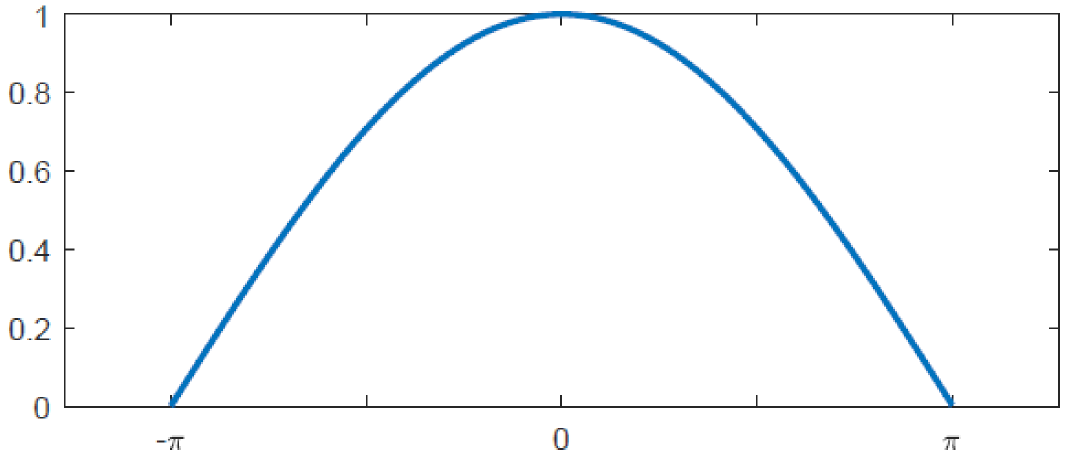
EXAMPLE: AVERAGE FILTER

- $y[n] = \frac{1}{2}(x[n] + x[n-1])$

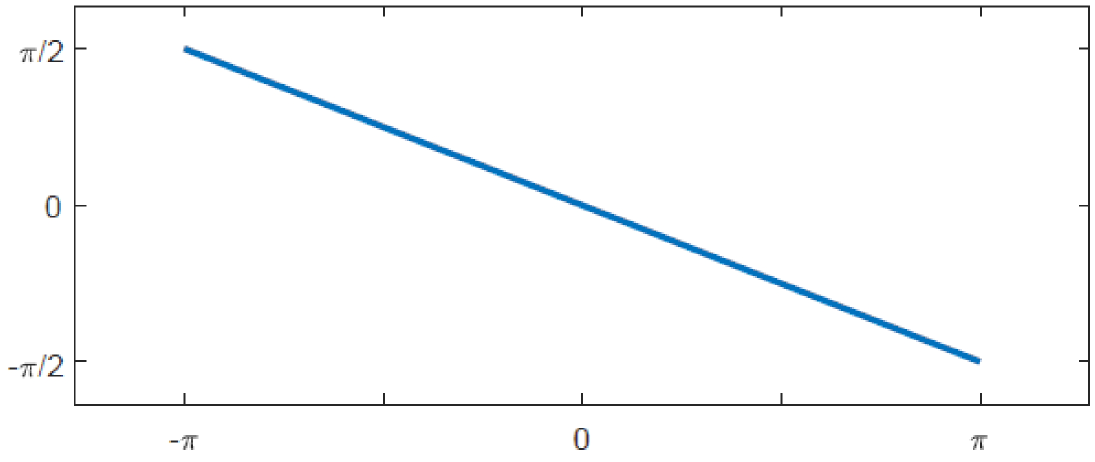
$$h[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right) \qquad \longleftrightarrow \qquad H(e^{j\omega}) = \frac{1}{2}\left[1 + e^{-j\omega}\right]$$

$$\underbrace{\cos\left(\frac{\omega}{2}\right)}_{|H(e^{j\omega})|}\ \underbrace{e^{-j\omega/2}}_{\angle H(e^{j\omega})}$$

- Low-pass filter used for smoothing



(a) $|H(e^{j\omega})| = \cos(\omega/2)$



(b) $\angle H(e^{j\omega}) = -\omega/2$

MATLAB FOR FILTERS

- Very helpful to visualize filters

```
w = -pi:0.01:pi;                    %define freq range
H = cos(w/2) .* exp(-j*(w/2));
figure, plot(w, abs(H))
figure, plot(w, phase(H))           %or use angle(H)
```

FOURIER SERIES SUMMARY

- Continuous Case
- $x(t) = \sum_k a_k e^{jk\omega_0 t}$
- $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$
- Fundamental frequency ω_0
- Fundamental period $T = \frac{2\pi}{\omega_0}$

- Discrete Case
- $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$
- $a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\omega_0 n}$
- Fundamental frequency ω_0
- Fundamental period $N = \frac{2\pi}{\omega_0}$