Homework #8  
Due Su 11/26

Note:  
OW Oppenheim and Wilsky  
SSS Schaum’s Signals and Systems  
SPR Schaum’s Probability, Random Variables, and Random Processes

Be sure to show all your work for credit.

1. (SPR 4.87)  
Let $Y = 2X + 3$. Find the pdf of $Y$ if $X$ is a uniform r.v. over $(-1, 2)$.

**Solution**

The solution can be found with the equation

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = f_X[h(y)] \left| \frac{dh(y)}{dy} \right| .$$

Since $h(y) = 0.5(y - 3)$, the solution is

$$f_Y(y) = f_X[h(y)] \left| \frac{dh(y)}{dy} \right|$$

$$= f_X(0.5(y - 3)) \frac{1}{2}$$

Remember that since $X \sim U[-1, 2]$, $f_X(x) = \frac{1}{3}$ is a constant value over the interval.

$$= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

Notice this is defined over the interval $x = [-1, 3]$. When the limits are plugged into $h(y)$ this results in $y = [1, 7]$, meaning $Y$ is a uniform random variable $Y \sim U[1, 7]$. See problem 4.2, 4.3, and 4.4 for more examples.

2. (SPR 4.92)  
Let $X$ denote the number of heads obtained when three independent tossings of a fair coin are made. Let $Y = X^2$. Find $E[Y]$ and $Var(Y)$.

**Solution**

Note that since this is a Binomial RV with distribution $X \sim B(3, 0.5)$, the probability can be found from $p_X(k) = \binom{3}{k} 0.5^k (1 - 0.5)^{3-k}$. This results in pmf

$$p_X(0) = \frac{1}{8} \quad p_X(1) = \frac{3}{8} \quad p_X(2) = \frac{3}{8} \quad p_X(3) = \frac{1}{8}.$$
The expected value can then be computed by definition

\[ E[Y] = E[X^2] = \sum_{i} x_i^2 p_X(x_i) \]

\[ = \sum_{k=0}^{3} k^2 p_X(k) \]

\[ = 0^2 p_X(0) + 1^2 p_X(1) + 2^2 p_X(2) + 3^2 p_X(3) \]

\[ = 1\left(\frac{3}{8}\right) + 4\left(\frac{3}{8}\right) + 9\left(\frac{1}{8}\right) \]

\[ = \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = 3 \]

Again, use definition for variance \( Var(Y) = E[Y^2] - E^2[Y] \). First compute second moment

\[ E[Y^2] = E[X^4] = \sum_{x_i} x_i^4 p_X(x_i) \]

\[ = \sum_{k=0}^{3} k^4 p_X(k) \]

\[ = 0^4 p_X(0) + 1^4 p_X(1) + 2^4 p_X(2) + 3^4 p_X(3) \]

\[ = 1\left(\frac{3}{8}\right) + 16\left(\frac{3}{8}\right) + 81\left(\frac{1}{8}\right) \]

\[ = \frac{3}{8} + \frac{48}{8} + \frac{81}{8} = 16.5 \]

Use second moment to compute variance.

\[ Var(Y) = E[Y^2] - E^2[Y] \]

\[ = 16.5 - 3^2 = 7.5 \]

3. (SPR 5.84)

Consider a random process \( X(t) \) defined by

\[ X(t) = Y \cos(\omega t + \Theta) \]

where \( Y \) and \( \Theta \) are independent r.v.’s and are uniformly distributed over \((-A, A)\) and \((-\pi, \pi)\) respectively.

(a) Find the mean of \( X(t) \).

(b) Find the autocorrelation function \( R_X(t, s) \) of \( X(t) \).

Hint: Be sure to look at Problem 5.20 to help on these problems.

**Solution**

(a) Note: this problem is very similar to the example problem from lecture.

\[ E[X(t)] = E[Y \cos(\omega t + \Theta)] \]

\[ = E_Y[Y] E_\Theta[\cos(\omega t + \Theta)] \] by independence

\[ = 0 \]
(b) Make use of lecture example and that $Y$ is a zero-mean RV.

\[
R_x(t,s) = E[X(t)X(s)] = E[Y \cos(\omega t + \Theta)Y \cos(\omega s + \Theta)]
\]
\[
= E[Y^2 \cos(\omega t + \Theta) \cos(\omega s + \Theta)]
\]
\[
= E_Y[Y^2]E_\Theta[\cos(\omega t + \Theta) \cos(\omega s + \Theta)]
\]
\[
= Var(Y) \left(\frac{1}{2} \cos \omega \tau\right) \quad \text{independence, zero-mean, lecture}
\]
\[
= \frac{4A^2}{12} \cos \omega (s-t)
\]
\[
= \frac{4A^2}{6} \cos \omega (s-t)
\]

4. (SPR 5.85)

Suppose that a random process $X(t)$ is wide-sense stationary with autocorrelation

\[
R_X(t, t + \tau) = e^{-|\tau|/2}.
\]

(a) Find the second moment of the r.v. $X(5)$.

(b) Find the second moment of the r.v. $X(5) - X(3)$.

Solution

(a)

\[
E[X^2(5)] = E[X(5)X(5)] = R_X(5, 5 + 0) = e^{-0/2} = 1
\]

(b)

\[
E[(X(5) - X(3))^2] = E[X^2(5) - 2X(5)X(3) + X^2(3)]
\]
\[
= E[X^2(5)] - 2E[X(5)X(3)] + E[X^2(3)]
\]
\[
= R_X(0) - 2R_X(2) + R_X(0)
\]
\[
= 2(R_X(0) - R_X(2))
\]
\[
= 2(1 - e^{-1})
\]

5. (SPR 5.87)

Consider the random processes

\[
X(t) = A_0 \cos(\omega_0 t + \Theta) \quad Y(t) = A_1 \cos(\omega_1 t + \Phi)
\]

where $A_0, A_1, \omega_0, \omega_1$ are constants and r.v.’s $\Theta$ and $\Phi$ are independent and uniformly distributed over $(-\pi, \pi)$.

(a) Find the cross-correlation function $R_{XY}(t, t + \tau)$ of $X(t)$ and $Y(t)$.

(b) Repeat (a) if $\Theta = \Phi$.

Solution
(a) \[ R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[A_0 \cos(\omega_0 t + \Theta)A_1 \cos(\omega_1(t+\tau) + \Phi)] \]
\[ = A_0 A_1 E[\cos(\omega_0 t + \Theta) \cos(\omega_1(t + \tau) + \Phi)] \]

Given independence \( f_{\Theta \Phi}(\theta, \phi) = f_{\Theta}(\theta) f_{\Phi}(\phi) = \frac{1}{2\pi} \frac{1}{2\pi} = \frac{1}{4\pi} \)
\[ = A_0 A_1 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) \cos(\omega_1(t+\tau) + \phi) \frac{1}{4\pi} d\phi d\theta \]
\[ = \frac{A_0 A_1}{4\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t + \phi) d\phi \int_{-\pi}^{\pi} \cos(\omega_1(t + \tau) + \phi) d\phi \]
\[ = 0 \]

(b) When \( \Theta = \Phi \) this is a single RV problem (similar to lecture).
\[ R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[A_0 \cos(\omega_0 t + \Theta)A_1 \cos(\omega_1 t + \Theta)] \]

Use trig identity \( \cos u \cos v = \frac{1}{2} [\cos(u - v) + \cos(u + v)] \)
\[ = \frac{A_0 A_1}{2} E[\cos(\omega_0 t + \Theta - \omega_1(t + \tau) - \Theta) \cos(\omega_0 t + \Theta + \omega_1(t + \tau) + \Theta)] \]
\[ = \frac{A_0 A_1}{2} E[\cos((\omega_0 - \omega_1)t - \omega_1(\tau)) + \cos((\omega_0 + \omega_1)t + 2\Theta + \omega_1\tau)] \]
\[ = \frac{A_0 A_1}{2} \cos((\omega_0 - \omega_1)t - \omega_1(\tau)) \]
\[ = \frac{A_0 A_1}{2} \cos((\omega_1 - \omega_0)t + \omega_1(\tau)) \]