Homework #5  
Due Th. 10/22

Note:  
OW  Oppenheim and Wilsky  
SSS  Schaum’s Signals and Systems  
SPR  Schaum’s Probability, Random Variables, and Random Processes (2nd edition)

Be sure to show all your work for credit.

1. (SPR 1.85)  
A random experiment has sample space \( S = \{a, b, c\} \). Suppose that \( P(\{a, c\}) = 0.75 \) and \( P(\{b, c\}) = 0.6 \). Find the probabilities of the elementary elements.

\textbf{Solution}  
Note that \( P(\{a, c\}) = P(a) + P(c) = 0.75 \) \( P(\{b, c\}) = P(b) + P(c) = 0.6 \)

The individual event probabilities can be found as  
\[
\begin{align*}
P(a) &= P(S) - P(\{b, c\}) = 1 - 0.6 = 0.4 \\
P(b) &= P(S) - P(\{a, c\}) = 1 - 0.75 = 0.25 \\
P(c) &= P(\{a, c\}) - P(a) = 0.75 - 0.4 = 0.35.
\end{align*}
\]

2. (SPR 1.86)  
Show that

(a) \( P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) \)

(b) \( P(A \cap B) \geq 1 - P(\bar{A}) - P(\bar{B}) \)

(c) \( P(A \Delta B) = P(A \cup B) - P(A \cap B) \)

\textbf{Solution}  
(a) Using De Morgan’s property,  
\[
P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B})
\]

By complement property  
\[
P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B).
\]

(b) Stating with the left-side of the inequality  
\[
P(A \cap B) = P(A) + P(B) - P(A \cup B)
\]

\[
= (1 - P(\bar{A})) + (1 - P(\bar{B})) - P(A \cup B)
\]

\[
= 1 - P(\bar{A}) - P(\bar{B}) + 1 - P(A \cup B)
\]

\[
\leq 1
\]

\[
\geq 0
\]

Therefore,  
\[
P(A \cap B) \geq 1 - P(\bar{A}) - P(\bar{B}).
\]

(c) This can easily be seen by examining Fig. 1-2(b) in the book.
3. (SPR 1.87)

Let $A, B,$ and $C$ be three events in $S$. If $P(A) = P(B) = \frac{1}{4}, P(C) = \frac{1}{3}, P(A \cap B) = \frac{1}{8}, P(A \cap C) = \frac{1}{6},$ and $P(B \cap C) = 0$, find $P(A \cup B \cup C)$.

**Solution**

Use a Venn Diagram to help visualize this.

$$P(A \cup B \cup C) = P(A \cup C) - P(A \cap C) + P(A \cup B) - P(A \cap B) - P(A)$$

$$= P(A) + P(C) - P(A \cap C) + P(B) - P(A \cap B)$$

$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8}$$

$$= \frac{13}{24}.$$

4. (SPR 1.90)

In an experiment consisting of 10 throws of a pair of fair dice, find the probability of the event that at least one double 6 occurs.

**Solution**

When throwing a pair of dice (each denoted as D1 and D2) there are six possible outcomes for each die resulting in $6^2 = 36$ different D1-D2 combinations. This results in a probability of $p_{66} = 1/36$ for rolling a double six. The event of getting a single double six is the complement of getting no double six at all. Therefore, the probability of getting at least a single double six in 10 rolls can be computed as

$$P_{10}(6 \times 6) = 1 - P_{no,66}^{10} = 1 - (1 - p_{66})^{10} = 1 - (1 - \frac{1}{36})^{10} = 0.2455.$$

5. (SPR 1.94)

An urn contains 8 white balls and 4 red balls. The experiment consisting of drawing 2 balls from the urn without replacement. Find the probability that both balls drawn are white.

**Solution**

Since there is no replacement after a ball draw, to solve this problem, analyze the situation after each ball draw separately. For the first ball draw (ball1), there are a total of 12 balls in the urn with 8 white and 4 red. Therefore, the probability of drawing a white ball on the first draw is

$$P(w1) = 8/12.$$

For ball2, there are now only 11 balls in the urn. Since the first ball taken was white, there are 7 white and 4 red resulting in

$$P(w2) = 7/11.$$

The total probability of drawing two white balls is

$$P(w1w2) = P(w1) \cdot P(w2) = \frac{8}{12} \cdot \frac{7}{11} = 0.4242.$$
6. (SPR 1.97)
Let $A$ and $B$ be two independent events in $S$. It is known that $P(A \cap B) = 0.16$ and $P(A \cup B) = 0.64$. Find $P(A)$ and $P(B)$.

Solution
First use the union probability to find

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Therefore,

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

$$= 0.64 + 0.16$$

$$= 0.8.$$

Now using the intersection probability and the independence constraint $P(A \cap B) = P(A)P(B)$

$$(0.8 - P(B))P(B) = 0.16$$

$$P^2(B) - 0.8P(B) + 0.16 = 0$$

$$(P(B) - 0.4)^2 = 0$$

This results in

$$P(A) = 0.4$$

$$P(B) = 0.4.$$

7. Let $A, B,$ and $C$ be events. Find expressions for the following events:

(a) Exactly one of the three events occurs.
(b) Exactly two of the events occur.
(c) One or more of the events occur.
(d) Two or more of the events occur.
(e) none of the events occur.

Solution
(a) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
(b) $(A \cap B \cap C^c) \cup (A^c \cap B \cap C) \cup (A^c \cap B \cap C)$
(c) $A \cup B \cup C$
(d) $(A \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C)$
(e) $A^c \cap B^c \cap C^c$
8. The number $U$ is selected at random from the unit interval. Let the events $A$ and $B$ be: $A =$ “$U$ differs from $1/2$ by more than $1/4$” and $B =$ “$1 - U$ is less than $1/2$”. Find the events:

(a) $A \cap B$
(b) $\tilde{A} \cap B$
(c) $A \cup B$

Solution
This problem is addressed by first understanding what each event means.

$A$:
\[ \left| U - \frac{1}{2} \right| > \frac{1}{4} \Rightarrow U > \frac{3}{4} \text{ or } U < \frac{1}{4} \]

$B$:
\[ 1 - U < \frac{1}{2} \Rightarrow U > \frac{1}{2} \]

It might be easiest to solve these by drawing out the events on two number lines.

(a) $A \cap B = \{ U > \frac{3}{4} \}$
(b) $\tilde{A} \cap B = (\frac{1}{4} \leq U \leq \frac{3}{4}) \& (U > \frac{1}{2}) = \{ \frac{1}{2} < U \leq \frac{3}{4} \}$
(c) $A \cup B = \{ U > \frac{1}{2} \text{ or } U < \frac{1}{4} \}$