Homework #2
Due Th. 9/10

Note: OW Oppenheim and Wilsky
SSS Schaum’s Signals and Systems
SPR Schaum’s Probability, Random Variables, and Random Processes

Be sure to show all your work for credit.

1. Determine the Fourier Series representation for each of the following signals if it exists.

(a) \( x(t) = \cos(3\pi t + \frac{\pi}{3}) \)

(b) \( x(t) = \sin^2(\pi t) \)

(c) \( x[n] = 2\cos(1.6\pi n) + \sin(2.4\pi n) \)

(d) \( x[n] = n \) for \( n = 0, 1, 2, 3 \) with period \( N = 4 \)

Solution

(a) Use Euler’s relationship with \( \omega_0 = 3\pi \) and \( T = \frac{2\pi}{\omega_0} = \frac{2}{3} \).

\[
x(t) = \cos \left( 3\pi t + \frac{\pi}{3} \right) = \frac{1}{2} \left[ e^{j(3\pi t + \pi/3)} + e^{-j(3\pi t + \pi/3)} \right] = \frac{1}{2} e^{j\pi/3} e^{j3\pi t} + \frac{1}{2} e^{-j\pi/3} e^{-j3\pi t}
\]

(b) Use trig identities or Euler’s to simplify. Notice the period is half of \( \sin(\pi t) \), \( T = 1 \) and \( \omega_0 = 2\pi \).

\[
x(t) = \sin^2(\pi t) = \frac{1 - \cos(2\pi t)}{2}
\]

\[
= \frac{1}{2} - \frac{1}{2} \left[ \frac{1}{2} (e^{j2\pi t} + e^{-j2\pi t}) \right] = \frac{1}{2} e^{j2\pi t} + \frac{1}{4} e^{-j2\pi t}
\]

(c) Find the period of each sinusoid and determine the period of \( x[n] \) as the least common multiple.

Use the discrete periodicity constraint \( \omega_0 N = 2\pi k \) to find smallest integers \( k, N \).

\[
x_1[n] = \cos(1.6\pi n) \quad \Rightarrow \quad N_1 = \frac{2\pi k}{1.6} = 5
\]

\[
x_2[n] = \cos(2.4\pi n) \quad \Rightarrow \quad N_2 = \frac{2\pi k}{2.4} = 5
\]
Therefore, the period of $x[n]$ is $T = 5$ and $\omega_0 = \frac{2\pi}{5}$.

\[
x[n] = 2 \left( \frac{1}{2} e^{j1.6\pi n} + \frac{1}{2} e^{-j1.6\pi n} \right) + \left( \frac{1}{2j} e^{j2.4\pi n} - \frac{1}{2j} e^{-j2.4\pi n} \right)
\]

\[
= \frac{1}{c_4} e^{j1.6\pi n} + \frac{1}{c_{-4}} e^{-j1.6\pi n} + \frac{1}{c_6} e^{j2.4\pi n} - \frac{1}{c_{-6}} e^{-j2.4\pi n}
\]

(d) This problem is easiest to solve by calculating the $N = 4$ coefficients individually.

\[
a_k = \frac{1}{N} \sum_{n=-N}^{N} x[n] e^{-j\frac{2\pi}{N} n}
\]

Applying the FS sum

\[
a_0 = \frac{1}{4} \sum_{n=0}^{3} n = \frac{3}{2}
\]

\[
a_1 = \frac{1}{4} \sum_{n=0}^{3} ne^{-j\frac{2\pi}{5} n}
= \frac{1}{4} \left[ -j - 2 + 3j \right] = \frac{1}{2} (-1 + j)
\]

\[
a_2 = \frac{1}{4} \sum_{n=0}^{3} ne^{-j\frac{4\pi}{5} n}
= \frac{1}{4} \left[ -1 + 2 + 3 \right] = \frac{1}{2}
\]

\[
a_3 = \frac{1}{4} \sum_{n=0}^{3} e^{-3j\frac{2\pi}{5} n}
= \frac{1}{4} \left[ -j - 2 + 3j \right] = \frac{1}{2} (-1 - j)
\]

2. (SSS 5.61 (a),(b))

Solution

(a) It is easy to find the period of $x(t)$ by noting that it is half of the non-absolute value version of the signal as can be seen in Fig. 1. Therefore,

\[
T = 1 \quad \omega_0 = 2\pi.
\]

(b) The FS coefficients are found by integrating over a single period.

\[
a_k = \frac{1}{T} \int_{T} x(t) e^{-j\omega_0 t} dt
\]

\[
= \int_{0}^{1} A \left[ e^{j\pi t} - e^{-j\pi t} \right] e^{-j2\pi t} dt
\]

\[
= A \int_{0}^{1} \left[ e^{j\pi (1-2k)t} - e^{-j\pi (1+2k)t} \right] dt
\]
\[ a_k = \frac{A}{2j} \left[ \frac{e^{j\pi(1-2k)t}}{j\pi(1-2k)} - \frac{e^{-j\pi(1+2k)t}}{-j\pi(1+2k)} \right]_0^1 \]

\[ = \frac{A}{2j} \left[ \frac{e^{j\pi(1-2k)} - 1}{j\pi(1-2k)} + \frac{e^{-j\pi(1+2k)} - 1}{j\pi(1+2k)} \right] \]

\[ = \frac{A}{j^22\pi} \left[ \frac{(e^{j\pi(1-2k)} - 1)(1 + 2k) + (e^{-j\pi(1+2k)} - 1)(1 - 2k)}{(1 - 2k)(1 + 2k)} \right] \]

Since \( e^{j\pi(1-2k)} = e^{j\pi} e^{-j2\pi k} = -1 \cdot 1 = -1, \)

\[ = \frac{A}{-2\pi(1 - 4k^2)} \left[ (-1 - 1)(1 + 2k) + (-1 - 1)(1 - 2k) \right] \]

\[ = \frac{A}{-2\pi(1 - 4k^2)} [-4] \]

\[ = \frac{2A}{\pi(1 - 4k^2)} \]

3. (OW 3.24)

Solution

(a) Find the average signal value over a period

\[ a_0 = \frac{1}{T} \int_T x(t)dt = \frac{1}{2} \left[ \frac{1}{2}(1)(2) \right] = \frac{1}{2}. \]

(b) Define

\[ g(t) = \frac{dx(t)}{dt} = \begin{cases} 1 & 0 \leq t \leq 1 \\ -1 & 1 \leq t \leq 2 \end{cases}. \]

Then FS coefficients can be found by noting

\[ g(t) \rightarrow b_k = \frac{1}{T} \int_T g(t)e^{-jk\omega_0 t}dt. \]
\(k = 0\) \quad \(b_0 = 0\) \quad \text{average over single period}

\(k \neq 0\) \quad b_k = \frac{1}{2} \left[ \int_{0}^{1} e^{-j k \omega_0 t} dt - \int_{1}^{2} e^{-j k \omega_0 t} dt \right]

\begin{align*}
&= \frac{1}{2} \left[ \frac{1}{-j k \omega_0} e^{-j k \omega_0} \right]_0^1 - \frac{1}{2} \left[ \frac{1}{-j k \omega_0} e^{-j k \omega_0} \right]_1^2 \\
&= \frac{1}{2 j k \omega_0} \left[ 1 - e^{-j \omega_0} \right] + \frac{1}{2 j k \omega_0} \left[ e^{-j 2 \omega_0} - e^{-j \omega_0} \right] \\
&= \frac{1}{2 j k \omega_0} \left[ 1 - 2 e^{-j \omega_0} + e^{-j 2 \omega_0} \right] \\
&= \frac{1}{2 j k \pi} \left[ 1 - e^{-j k \pi} + \frac{e^{-j 2 \pi}}{1} \right] \\
&= \frac{1}{j k \pi} \left[ 1 - e^{-j k \pi} \right]
\end{align*}

(c) Using the differentiation property

\[x(t) \leftrightarrow a_k\]
\[
\frac{dx(t)}{dt} \leftrightarrow b_k = j k \omega_0 a_k = j k \pi a_k
\]

\[a_k = \frac{1}{j k \pi} b_k = \frac{1}{(j k \pi)^2} \left[ 1 - e^{-j \pi} \right] = \frac{1}{k^2 \pi^2} \left[ e^{-j \pi} - 1 \right]
\]

4. (OW 3.30)

Solution

(a) Use Euler relationship and recognize that \(N = 6\) and \(\omega_0 = \frac{2\pi}{6}\).

\[x[n] = 1 + \cos \left( \frac{2\pi}{6} n \right)
\]

\[= \frac{1}{a_0} + \frac{1}{a_1} e^{j \omega_0 n} + \frac{1}{a_{-1}} e^{-j \omega_0 n}
\]

(b) Again, \(N = 6\) and \(\omega_0 = \frac{2\pi}{6}\).

\[y[n] = \sin \left( \frac{2\pi}{6} + \frac{\pi}{4} \right)
\]

\[= \frac{1}{2j} e^{j (\omega_0 n+\pi/4)} - \frac{1}{2j} e^{-j (\omega_0 n+\pi/4)}
\]

\[= \frac{1}{2j} e^{j \pi/4} e^{j \omega_0 n} + \frac{1}{2j} e^{-j \pi/4} e^{-j \omega_0 n}
\]

(c) Using the multiplication property, \(z[n] = x[n]y[n] \leftrightarrow c_k = \sum_{l=-N}^{N} a_l b_{k-l} = a_k * b_k\). There are various ways to solve this but it is helpful to rewrite the FS coefficients as impulses and do the convolution with the resulting signals.

\[c_k = (a_{-1} \delta[k + 1] + a_0 \delta[k] + a_1 \delta[k - 1]) \ast (b_{-1} \delta[k + 1] + b_1 \delta[k - 1])
\]
\[ = a_{-1} b_{-1} \delta[k + 2] + a_{-1} b_1 \delta[k] + a_0 b_{-1} \delta[k + 1] + a_0 b_1 \delta[k - 1] + a_1 b_{-1} \delta[k] + a_1 b_1 \delta[k - 2] \]

The coefficients can be found as
\[
c_{-2} = a_{-1} b_{-1} = -\left( \frac{1}{2} \right) \frac{1}{2^j} e^{-j\pi/4} \quad c_2 = a_1 b_1 = \left( \frac{1}{2} \right) 2^j e^{j\pi/4}
\]
\[
c_{-1} = a_0 b_{-1} = -\frac{1}{2} e^{-j\pi/4} \quad c_1 = a_0 b_1 = \frac{1}{2^j} e^{j\pi/4}
\]
\[
c_0 = a_{-1} b_1 + a_1 b_{-1} = \left( \frac{1}{2} \right) \frac{1}{2^j} e^{j\pi/4} - \left( \frac{1}{2} \right) \frac{1}{2^j} e^{-j\pi/4} = \frac{1}{2} \sin \left( \frac{\pi}{4} \right)
\]

(d) Solving this directly by Euler’s expansion results in the same answer. Below is a highlight of the answer using \( \omega_0 = \frac{2\pi}{6} \) and \( \theta = \frac{\pi}{4} \).

\[
z[n] = x[n] y[n] = \left( 1 + \cos \left( \frac{2\pi}{6} n \right) \right) \left( \sin \left( \frac{2\pi}{6} + \frac{\pi}{4} \right) \right)
\]
\[
= \sin \left( \frac{2\pi}{6} + \frac{\pi}{4} \right) + \sin \left( \frac{2\pi}{6} + \frac{\pi}{4} \right) \cos \left( \frac{2\pi}{6} n \right)
\]
\[
= \frac{1}{2^j} \left[ e^{j(\omega_0 + \theta)} - e^{-j(\omega_0 + \theta)} \right] + \frac{1}{4^j} \left[ e^{j(\omega_0 + \theta)} - e^{-j(\omega_0 + \theta)} \right] \left[ e^{j\omega_0} + e^{-j\omega_0} \right]
\]
\[
= \frac{1}{2^j} \left[ e^{j(\omega_0 + \theta)} - e^{-j(\omega_0 + \theta)} \right] + \frac{1}{4^j} \left[ e^{j(2\omega_0 + \theta)} + e^{j\theta} - e^{-j\theta} - e^{-j(2\omega_0 + \theta)} \right]
\]

gives \( c_{11} \) coefficients  
gives \( c_{12} \) and \( c_0 \) coefficients

5. (OW 3.31)

Solution

(a) See Fig. 2
(b) Use the FS equation:

\[ a_k = \frac{1}{N} \sum_{n=-N}^{N} x[n] e^{-jk\omega_0 n} \]

\[ b_k = \frac{1}{10} \sum_{n=0}^{9} g[n] e^{-jk\omega_0 n} \]

\[ = \frac{1}{10} \left[ 1 \cdot e^{-jk\omega_0(0)} + -1 \cdot e^{-jk\omega_0(8)} \right] \]

\[ = \frac{1}{10} \left[ 1 - e^{-j8\pi/5} \right] \]

(c) Using the first difference relationship

\[ g[n] = x[n] - x[n-1] \leftrightarrow b_k = (1 - e^{-jk\omega_0})a_k. \]

Therefore, the coefficients for \( x[n] \) are found as

\[ a_k = \frac{b_k}{1 - e^{-jk\omega_0}} = \frac{1}{10} \left[ 1 - e^{-j8\pi/5} \right]. \]

6. (OW 3.34)

**Solution**

In order to solve this problem, recognize that for an LTI system

\[ y(t) \leftrightarrow b_k = a_k H(jk\omega_0). \]

This requires finding

\[ H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt \]

\[ = \int_{-\infty}^{0} e^{4t}e^{-j\omega t} dt + \int_{0}^{\infty} e^{-4t}e^{-j\omega t} dt \]

\[ = \int_{-\infty}^{0} e^{(j\omega-4)t} dt + \int_{0}^{\infty} e^{-j(\omega+4)t} dt \]

\[ = \left[ \frac{e^{(j\omega-4)t}}{j\omega-4} \right]_{0}^{\infty} + \left[ \frac{e^{-j(\omega+4)t}}{-j(\omega+4)} \right]_{0}^{\infty} \]

\[ = \frac{1}{4-j\omega} + \frac{1}{4+j\omega} \]

(a) The impulse train has constant values for the FS coefficients and the period is \( T = 1 \) and \( \omega = 2\pi \).

\[ x(t) = \sum_{-\infty}^{\infty} \delta(t-n) \leftrightarrow a_k = 1 \]

\[ y(t) \leftrightarrow b_k = a_k H(jk\omega_0) = \frac{1}{4-jk\omega_0} + \frac{1}{4+jk\omega_0} \]
(b) Notice this signal has two samples (one positive and one negative), \( T = 2 \) and \( \omega_0 = \pi \). The signal can be re-written as

\[
x(t) = \sum_n \delta(t - 2n) - \sum_n \delta(t - 2n - 1)
\]

As in part (a), we know \( g(t) \leftrightarrow c_k = 1/T = 1/2 \). The FS of \( x(t) \) can be found using properties

\[
a_k = c_k - c_k e^{-jk\omega_0} = c_k (1 - e^{-jk\pi})
\]

\[
= \frac{1}{2} (1 - (-1)^k)
\]

\[
= \begin{cases} 
1 & k \text{ odd} \\
0 & k \text{ even}
\end{cases}
\]

The output FS coefficients are then

\[
b_k = a_k H(jk\omega_0) = \begin{cases} 
\frac{1}{4 - jk\pi} + \frac{1}{4 + jk\pi} & k \text{ odd} \\
0 & k \text{ even}
\end{cases}
\]

(c) In a similar process to part (b), find FS coefficients \( a_k \) to determine output FS \( b_k \). The square wave has \( T = 1 \) and \( \omega_0 = 2\pi \). Example 3.5 with \( T_1 = \frac{1}{4} \) should be used to find \( a_k \).

\[
a_k = \begin{cases} 
\frac{1}{2} & k = 0 \\
\sin(k\omega_0 T_1) & k \neq 0 \\
\frac{1}{2} \sin(k\pi/2) & k \neq 0
\end{cases}
\]

\[
b_k = \begin{cases} 
\frac{1}{2} \sin(k\pi/2) & k = 0 \\
\left[ \frac{1}{4 - jk2\pi} + \frac{1}{4 + jk2\pi} \right] & k \neq 0
\end{cases}
\]

7. (OW 3.35)

**Solution**

Given input signal \( x(t) \leftrightarrow a_k \) the output from an LTI system can be found as \( y(t) \leftrightarrow b_k = a_k H(jk\omega_0) \). Here we are given

\[ T = \frac{\pi}{7} \quad \omega_0 = \frac{2\pi}{T} = 14. \]

Since, we are told that \( y(t) = x(t) \) this implies that \( b_k = a_k \) which in turn implies that any \( a_k \) coefficient outside of the LTI cutoff frequency must be zero. Therefore, \( a_k = 0 \) when

\[ \omega_0 k \leq 250 \Rightarrow k \leq \frac{250}{14} = 17.86 \rightarrow 17. \]

8. (OW 3.62)

**Solution**
(a) It is easy to find the period of $y(t)$ by noting that it is half of the non-absolute value version of the signal $x(t)$ as can be seen in Fig. 3. Therefore, $T_x = 2\pi$ and $T_y = \pi$.

(b) This problem can be solved in the same way was was done for Problem 2 (SSS 5.61).

\[ b_k = \frac{2(-1)^k}{\pi(1 - 4k^2)} \]

(c) The DC component of a signal is the zeroth FS coefficient – the average of the signal over a single period.

\[ a_0 = \frac{1}{2\pi} \int_0^{2\pi} \cos t \, dt = 0 \]

\[ b_0 = \frac{1}{\pi} \int_T |\cos t| \, dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos t \, dt \]
\[ = \frac{1}{\pi} \left[ \sin t \right]_{-\pi/2}^{\pi/2} = \frac{1}{\pi} \left[ 1 - (-1) \right] \]
\[ = \frac{2}{\pi} \]