Homework #2
Due Su 9/14

Note:

OW	Oppenheim and Wilsky
SSS	Schaum's Signals and Systems
SPR	Schaum's Probability, Random Variables, and Random Processes

Be sure to show all your work for credit.

1. Determine the Fourier Series representation for each of the following signals if it exists.

   (a) $x(t) = \cos\left(3\pi t + \dfrac{\pi}{3}\right)$

   (b) $x(t) = \sin^2(\pi t)$

   (c) $x[n] = 2\cos(1.6\pi n) + \sin(2.4\pi n)$

   (d) $x[n] = n$ for $n = 0, 1, 2, 3$ with period $N = 4$

   Solution

   (a) Use Euler's relationship with $\omega_0 = 3\pi$ and $T = \frac{2\pi}{\omega_0} = \frac{2}{3}$.

$$
\begin{aligned}
x(t) = \cos\left(3\pi t + \frac{\pi}{3}\right) &= \frac{1}{2}\left[e^{j(3\pi t+\pi/3)} + e^{-j(3\pi t+\pi/3)}\right] \\
&= \underbrace{\frac{1}{2}e^{j\pi/3}}_{a_1} e^{j3\pi t} + \underbrace{\frac{1}{2}e^{-j\pi/3}}_{a_{-1}} e^{-j3\pi t}
\end{aligned}
$$

   (b) Use trig identities or Euler's to simplify. Notice the period is half of $\sin(\pi t)$, $T = 1$ and $\omega_0 = 2\pi$.

$$
\begin{aligned}
x(t) = \sin^2(\pi t) &= \frac{1 - \cos(2\pi t)}{2} \\
&= \frac{1}{2} - \frac{1}{2}\left[\frac{1}{2}\left(e^{j2\pi t} + e^{-j2\pi t}\right)\right] \\
&= \underbrace{\frac{1}{2}}_{a_0} + \underbrace{-\frac{1}{4}}_{a_1} e^{j2\pi t} + \underbrace{-\frac{1}{4}}_{a_{-1}} e^{-j2\pi t}
\end{aligned}
$$

   (c) Find the period of each sinusoid and determine the period of $x[n]$ as the least common multiple.

   Use the discrete periodicity constraint $\omega_0 N = 2\pi k$ to find smallest integers k, N.

$$
\begin{aligned}
x_1[n] &= \cos(1.6\pi n) \qquad &\Rightarrow N_1 = \frac{2\pi k}{1.6\pi} = 5 \\
x_2[n] &= \cos(2.4\pi n) \qquad &\Rightarrow N_2 = \frac{2\pi k}{2.4\pi} = 5
\end{aligned}
$$

Therefore, the period of $x[n]$ is $N = 5$ and $\omega_0 = \frac{2\pi}{5}$.

$$x[n] = 2\left[\frac{1}{2}e^{j1.6\pi n} + \frac{1}{2}e^{-j1.6\pi n}\right] + \left[\frac{1}{2j}e^{j2.4\pi n} - \frac{1}{2j}e^{-j2.4\pi n}\right]$$
$$= \underbrace{1}_{c_4} e^{j1.6\pi n} + \underbrace{1}_{c_{-4}} e^{-j1.6\pi n} + \underbrace{\frac{1}{2j}}_{c_6} e^{j2.4\pi n} + \underbrace{-\frac{1}{2j}}_{c_{-6}} e^{-j2.4\pi n}$$

However, note the period is $N = 5$ which means that there are only 5 unique coefficients and the c_6 coefficient is actually a wrap from a lower frequency (w_0). This results in coefficients from a single period as:

$$x[n] = \underbrace{(1 - 0.5j)}_{a_1} e^{jw_0 n} + \underbrace{(1 + 0.5j)}_{a_4} e^{j4w_0 n}$$

(d) This problem is easiest to solve by calculating the $N = 4$ coefficients individually.

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n}$$

Applying the FS sum

$$a_0 = \frac{1}{4}\sum_{n=0}^{3} n = \frac{3}{2}$$
$$a_1 = \frac{1}{4}\sum_{n=0}^{3} ne^{-j\frac{\pi}{2}n} = \frac{1}{4}\left[e^{-j\frac{\pi}{2}} + 2e^{-j\frac{\pi}{2}2} + 3e^{-j\frac{\pi}{2}3}\right]$$
$$= \frac{1}{4}[-j - 2 + 3j] = \frac{1}{2}(-1 + j)$$
$$a_2 = \frac{1}{4}\sum_{n=0}^{3} ne^{-j\pi n} = \frac{1}{4}\left[e^{-j\pi} + 2e^{-j2\pi} + 3e^{-j3\pi}\right]$$
$$= \frac{1}{4}[-1 + 2 + 3] = -\frac{1}{2}$$
$$a_3 = \frac{1}{4}\sum_{n=0}^{3} ne^{-j3\frac{\pi}{2}n} = \frac{1}{4}\left[e^{-j3\frac{\pi}{2}} + 2e^{-j3\frac{\pi}{2}2} + 3e^{-j3\frac{\pi}{2}3}\right]$$
$$= \frac{1}{4}[j - 2 - 3j] = \frac{1}{2}(-1 - j)$$

2. (OW 3.24)

Solution

(a) Find the average signal value over a period

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{2}\left[\frac{1}{2}(1)(2)\right] = \frac{1}{2}.$$

(b) Define

$$g(t) = \frac{dx(t)}{dt} = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 \le t \le 2 \end{cases}.$$

Then FS coefficients can be found by noting

$$g(t) \longrightarrow b_k = \frac{1}{T}\int_T g(t)e^{-jk\omega_0 t}dt.$$

$$\begin{aligned}
k = 0 \qquad & b_0 = 0 \qquad \text{average over single period} \\
k \neq 0 \qquad & b_k = \frac{1}{2}\left[\int_0^1 e^{-jk\omega_0 t}dt - \int_1^2 e^{-jk\omega_0 t}dt\right] \\
&= \frac{1}{2}\left[\frac{1}{-jk\omega_0}e^{-jk\omega_0}\right]_0^1 - \frac{1}{2}\left[\frac{1}{-jk\omega_0}e^{-jk\omega_0}\right]_1^2 \\
&= \frac{1}{2jk\omega_0}\left[1 - e^{-jk\omega_0}\right] + \frac{1}{2jk\omega_0}\left[e^{-jk2\omega_0} - e^{-jk\omega_0}\right] \\
&= \frac{1}{2jk\omega_0}\left[1 - 2e^{-jk\omega_0} + e^{-jk2\omega_0}\right] \\
&= \frac{1}{2jk\pi}\left[1 - 2e^{-jk\pi} + \underbrace{e^{-jk2\pi}}_{=1}\right] \\
&= \frac{1}{jk\pi}\left[1 - e^{-jk\pi}\right]
\end{aligned}$$

(c) Using the differentiation property

$$\begin{aligned}
x(t) &\longleftrightarrow a_k \\
\frac{dx(t)}{dt} &\longleftrightarrow b_k = jk\omega_0 a_k = jk\pi a_k \\
&\Rightarrow a_k = \frac{1}{jk\pi}b_k = \frac{1}{(jk\pi)^2}[1 - e^{-j\pi}] = \frac{1}{k^2\pi^2}[e^{-j\pi} - 1]
\end{aligned}$$

Combining this with part (a) the final FS representation is

$$a_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{k^2\pi^2}[e^{-j\pi} - 1] & k \neq 0 \end{cases}.$$

3. (OW 3.30)

Solution

(a) Use Euler relationship and recognize that $N = 6$ and $\omega_0 = \frac{2\pi}{6}$.

$$\begin{aligned}
x[n] &= 1 + \cos\left(\frac{2\pi}{6}n\right) \\
&= \underbrace{1}_{a_0} + \underbrace{\frac{1}{2}}_{a_1} e^{j\omega_0 n} + \underbrace{\frac{1}{2}}_{a_{-1}} e^{-j\omega_0 n}
\end{aligned}$$

(b) Again, $N = 6$ and $\omega_0 = \frac{2\pi}{6}$.

$$\begin{aligned}
y[n] &= \sin\left(\frac{2\pi}{6} + \frac{\pi}{4}\right) \\
&= \frac{1}{2j}e^{j(\omega_0 n + \pi/4)} - \frac{1}{2j}e^{-j(\omega_0 n + \pi/4)}
\end{aligned}$$

$$= \underbrace{\frac{1}{2j}e^{j\pi/4}}_{b_1} e^{j\omega_0 n} + \underbrace{-\frac{1}{2j}e^{-j\pi/4}}_{b_{-1}} e^{-j\omega_0 n}$$

(c) Using the multiplication property, $z[n] = x[n]y[n] \longleftrightarrow c_k = \sum_{l=<N>} a_l b_{k-l} = a_k * b_k$. There are various ways to solve this but it is helpful to rewrite the FS coefficients as impulses and do the convolution with the resulting signals.

$$\begin{aligned} c_k &= (a_{-1}\delta[k+1] + a_0\delta[k] + a_1\delta[k-1]) * (b_{-1}\delta[k+1] + b_1\delta[k-1]) \\ &= a_{-1}b_{-1}\delta[k+2] + a_{-1}b_1\delta[k] + a_0 b_{-1}\delta[k+1] + a_0 b_1 \delta[k-1] + a_1 b_{-1}\delta[k] + a_1 b_1 \delta[k-2] \end{aligned}$$

The coefficients can be found as

$$c_{-2} = a_{-1}b_{-1} = -\left(\frac{1}{2}\right)\frac{1}{2j}e^{-j\pi/4} \qquad c_2 = a_1 b_1 = \left(\frac{1}{2}\right)2je^{j\pi/4}$$

$$c_{-1} = a_0 b_{-1} = -\frac{1}{2}e^{-j\pi/4} \qquad c_1 = a_0 b_1 = \frac{1}{2j}e^{j\pi/4}$$

$$c_0 = a_{-1}b_1 + a_1 b_{-1} = \left(\frac{1}{2}\right)\frac{1}{2j}e^{j\pi/4} - \left(\frac{1}{2}\right)\frac{1}{2j}e^{-j\pi/4} = \frac{1}{2}\sin\left(\frac{\pi}{4}\right)$$

(d) Solving this directly by Euler's expansion results in the same answer. Below is a highlight of the answer using $\omega_0 = \frac{2\pi}{6}$ and $\theta = \frac{pi}{4}$.

$$\begin{aligned} z[n] = x[n]y[n] &= \left(1 + \cos\left(\frac{2\pi}{6}n\right)\right)\left(\sin\left(\frac{2\pi}{6} + \frac{\pi}{4}\right)\right) \\ &= \sin\left(\frac{2\pi}{6} + \frac{\pi}{4}\right) + \sin\left(\frac{2\pi}{6} + \frac{\pi}{4}\right)\cos\left(\frac{2\pi}{6}n\right) \\ &= \frac{1}{2j}\left[e^{j(\omega_0+\theta)} - e^{-j(\omega_0+\theta)}\right] + \frac{1}{4j}\left[e^{j(\omega_0+\theta)} - e^{-j(\omega_0+\theta)}\right]\left[e^{j\omega_0} + e^{-j\omega_0}\right] \\ &= \underbrace{\frac{1}{2j}\left[e^{j(\omega_0+\theta)} - e^{-j(\omega_0+\theta)}\right]}_{\text{gives } c_{|1|} \text{ coefficients}} + \underbrace{\frac{1}{4j}\left[e^{j(2\omega_0+\theta)} + e^{j\theta} - e^{-j\theta} - e^{-j(2\omega_0+\theta)}\right]}_{\text{gives } c_{|2|} \text{ and } c_0 \text{ coefficients}} \end{aligned}$$

4. (OW 3.31)

Solution

(a) See Fig. 1.

(b) Use the FS equation:

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n} \\ b_k &= \frac{1}{10}\sum_{n=0}^{9} g[n]e^{-jk\omega_0 n} \\ &= \frac{1}{10}\left[1 \cdot e^{-jk\omega_0(0)} + -1 \cdot e^{-jk\omega_0(8)}\right] \\ &= \frac{1}{10}\left[1 - e^{-jk\frac{8\pi}{5}}\right] \end{aligned}$$

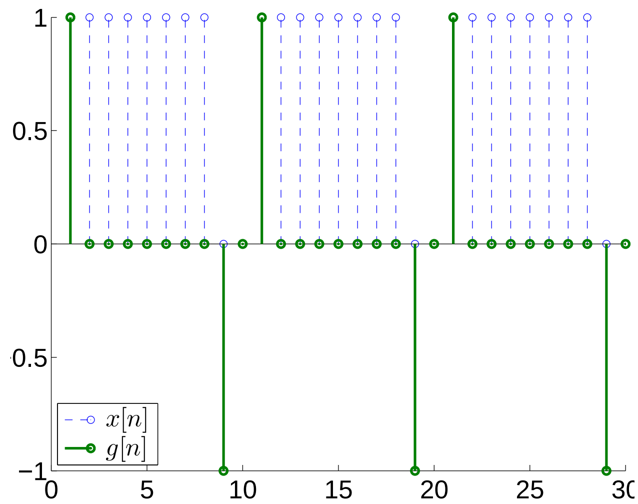


Figure 1: OW3.31(a)

(c) Using the first difference relationship

$$g[n] = x[n] - x[n-1] \longleftrightarrow b_k = (1 - e^{-jk\omega_0})a_k.$$

Therefore, the coefficients for $x[n]$ are found as

$$a_k = \frac{b_k}{1 - e^{-jk\omega_0}} = \frac{\frac{1}{10}\left[1 - e^{-jk\frac{8\pi}{5}}\right]}{1 - e^{-jk\frac{\pi}{5}}}.$$

5. (OW 3.34)

Solution

In order to solve this problem, recognize that for an LTI system

$$y(t) \longleftrightarrow b_k = a_k H(jk\omega_0).$$

This requires finding

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{0} e^{4t} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-4t} e^{-j\omega t} dt \\ &= \int_{0}^{\infty} e^{(j\omega - 4)t} dt + \int_{0}^{\infty} e^{-j(\omega+4)t} dt \\ &= \left[\frac{e^{(j\omega-4)t}}{j\omega - 4}\right]_0^{\infty} + \left[\frac{e^{-(j\omega+4)t}}{-(j\omega+4)}\right]_0^{\infty} \\ &= \frac{1}{4 - j\omega} + \frac{1}{4 + j\omega} \end{aligned}$$

(a) The impulse train has constant values for the FS coefficients and the period is $T = 1$ and $\omega_0 = 2\pi$.

$$x(t) = \sum_{-\infty}^{\infty} \delta(t - n) \longleftrightarrow a_k = 1$$

$$y(t) \leftrightarrow b_k = a_k H(jk\omega_0) = \frac{1}{4 - jk\omega_0} + \frac{1}{4 + jk\omega_0}$$

(b) Notice this signal has two samples (one positive and one negative), $T = 2$ and $\omega_0 = \pi$. The signal can be re-written as

$$x(t) = \underbrace{\sum_n \delta(t - 2n)}_{g(t)} - \underbrace{\sum_n \delta(t - 2n - 1)}_{g(n-1)}$$

As in part (a), we know $g(t) \leftrightarrow c_k = 1/T = 1/2$. The FS of $x(t)$ can be found using properties

$$\begin{aligned} a_k &= c_k - c_k e^{-jk\omega_0} = c_k(1 - e^{-jk\pi}) \\ &= \frac{1}{2}(1 - (-1)^k) \\ &= \begin{cases} 1 & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

The output FS coefficients are then

$$b_k = a_k H(jk\omega_0) = \begin{cases} \frac{1}{4 - jk\pi} + \frac{1}{4 + jk\pi} & k \text{ odd} \\ 0 & k \text{ even.} \end{cases}$$

(c) In a similar process to part (b), find FS coefficients a_k to determine output FS b_k. The square wave has $T = 1$ and $\omega_0 = 2\pi$. Example 3.5 with $T_1 = \frac{1}{4}$ should be used to find a_k.

$$\begin{aligned} a_k &= \begin{cases} \frac{1}{2} & k = 0 \\ \frac{\sin(k\omega_0 T_1)}{k\pi} & k \neq 0 \end{cases} \\ &= \begin{cases} \frac{1}{2} & k = 0 \\ \frac{\sin(k\pi/2)}{k\pi} & k \neq 0 \end{cases} \\ b_k &= \begin{cases} \frac{1}{2} & k = 0 \\ \frac{\sin(k\pi/2)}{k\pi} \left[\frac{1}{4 - jk2\pi} + \frac{1}{4 + jk2\pi}\right] & k \neq 0 \end{cases} \end{aligned}$$

6. (OW 3.35)

Solution

Given input signal $x(t) \longleftrightarrow a_k$ the output from an LTI system can be found as $y(t) \longleftrightarrow b_k = a_k H(jk\omega_0)$. Here we are given

$$T = \frac{\pi}{7} \qquad \omega_0 = \frac{2\pi}{T} = 14.$$

Since, we are told that $y(t) = x(t)$ this implies that $b_k = a_k$ which in turn implies that any a_k coefficient outside of the LTI cutoff frequency must be zero. Therefore, $a_k = 0$ when

$$\omega_0 k \leq 250 \Rightarrow k \leq \frac{250}{14} = 17.86 \to 17.$$

3.36 $y[n] - \frac{1}{4} y[n-1] = x[n]$

$$Y(z)\left[1 - \frac{1}{4}z^{-1}\right] = X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{4}z^{-1}}$$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

a) $x[n] = \sin\left(\frac{3\pi}{4}n\right)$ $\quad \omega_1 = \frac{3\pi}{4}, \; N = \frac{2\pi k}{\omega_1} = \frac{8}{3}k \Rightarrow 8, \; \omega_0 = \frac{2\pi}{N} = \frac{\pi}{4}$

$$= \underbrace{\frac{1}{2j}}_{a_3} e^{j\frac{3\pi}{4}n} - \underbrace{\frac{1}{2j}}_{a_{-3}} e^{-j\frac{3\pi}{4}n}$$

periodic with period 8

$$y[n] \leftrightarrow b_k = a_k H(e^{jk\omega_0}) = \begin{cases} \frac{1}{2j}\frac{1}{1-\frac{1}{4}e^{j3\pi/4}} & k = 3 \\ -\frac{1}{2j}\frac{1}{1-\frac{1}{4}e^{-j3\pi/4}} & k = -3 \end{cases}$$

periodic 8

b) $x[n] = \cos\left(\frac{\pi}{4}n\right) + 2\cos\left(\frac{\pi}{2}n\right)$

$\omega_1 = \frac{\pi}{4}$ $\quad \omega_2 = \frac{\pi}{2}$ $\quad N = \text{lcm}(8,4) = 8$

$N_1 = \frac{2\pi}{\pi/4} = 8$ $\quad N_2 = \frac{2\pi}{\pi/2} = 4$ $\quad \omega_0 = \frac{2\pi}{N} = \frac{\pi}{4}$

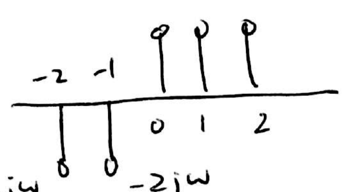
$$x[n] = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n} + e^{j2\omega_0 n} + e^{-j2\omega_0 n}$$

$$a_k = \begin{cases} \frac{1}{2} & k = \pm 1 \\ 1 & k = \pm 2 \end{cases} \Longleftrightarrow b_k = a_k H(e^{jk\omega_0}) = \begin{cases} \frac{1}{2}\frac{1}{1-\frac{1}{4}e^{j\pi/4}} & b_1 = b_{-1}^* \\ \frac{1}{1-\frac{1}{4}e^{j\pi/2}} & b_2 = b_{-2}^* \end{cases}$$

periodic with period 8

3.38

$$h[n] = \begin{cases} 1 & 0 \le n \le 2 \\ -1 & -2 \le n \le -1 \\ 0 & \text{else} \end{cases}$$



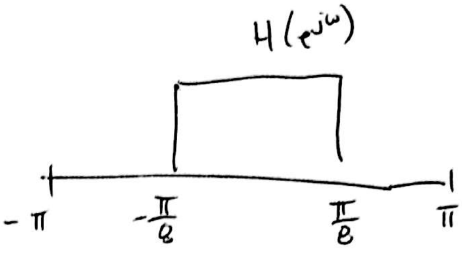
$$H(e^{j\omega}) = -e^{2j\omega} - e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}$$

$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$ impulse train $N = 4$, $\omega_0 = \frac{\pi}{2}$ $\longleftrightarrow a_k = \frac{1}{N} = \frac{1}{4}$

$$b_k = a_k H(e^{jk\omega_0}) = \frac{1}{4}\left(\underbrace{-e^{j\pi k}}_{(-1)^k} - e^{j\frac{\pi}{2}k} + 1 + e^{-j\frac{\pi}{2}k} + \underbrace{e^{-j\pi k}}_{(-1)^k}\right) = \frac{1}{4}\left(1 - e^{j\frac{\pi}{2}k} + e^{j\pi/2 k}\right)$$

3.39

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \frac{\pi}{8} \\ 0 & \frac{\pi}{8} < |\omega| < \pi \end{cases}$$



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$x[n]$ has period 3 $\Rightarrow N = 3$, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$

$k = 0$ $\quad k\omega_0 = 0$

$k = 1$ $\quad k\omega_0 = \frac{2\pi}{3}$

$k = 2$ $\quad k\omega_0 = \frac{2\pi}{3} \cdot 2 = \frac{4\pi}{3}$

first harmonic $\frac{2\pi}{3} >$ cutoff freq. $\omega_c = \frac{\pi}{8}$

$\Rightarrow$ only $k=0$ harmonic survives.

10. (OW 3.62)

Solution

(a) It is easy to find the period of $y(t)$ by noting that it is half of the non-absolute value version of the signal $x(t)$ as can be seen in Fig. 2. Therefore,

$$T_x = 2\pi \qquad T_y = \pi.$$

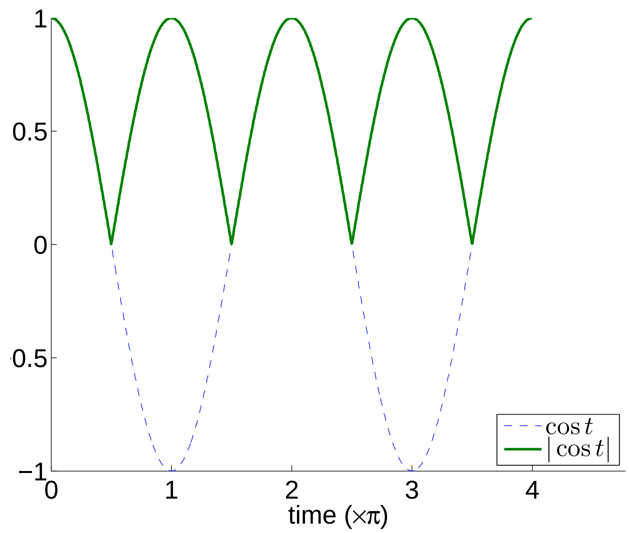

Figure 2: OW 3.62a

(b) This problem can be solved in the same way was was done for SSS 5.61 for a *sin* between $[0, 1]$. The FS coefficients are found by integrating over a single period with $A = 1$ and applying a time shift of $x(t-0.5) \leftrightarrow a_k e^{jk2\pi*0.5} = (-1)^k a_k$. Another approach is to integrate the *cos* between $[0.5, 1.5]$.

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \int_0^1 \frac{A}{2j}\left[e^{j\pi t} - e^{-j\pi t}\right]e^{-jk2\pi t}dt \\ &= \frac{A}{2}\int_0^1 \left[e^{j\pi(1-2k)t} - e^{-j\pi(1+2k)t}\right]dt \end{aligned}$$

$$\begin{aligned} a_k &= \frac{A}{2j}\left[\frac{e^{j\pi(1-2k)t}}{j\pi(1-2k)} - \frac{e^{-j\pi(1+2k)t}}{-j\pi(1+2k)}\right]_0^1 \\ &= \frac{A}{2j}\left[\frac{e^{j\pi(1-2k)}-1}{j\pi(1-2k)} + \frac{e^{-j\pi(1+2k)}-1}{j\pi(1+2k)}\right] \\ &= \frac{A}{j^2 2\pi}\left[\frac{(e^{j\pi(1-2k)}-1)(1+2k) + (e^{-j\pi(1+2k)}-1)(1-2k)}{(1-2k)(1+2k)}\right] \end{aligned}$$

Since $e^{j\pi(1-2k)} = e^{j\pi}e^{-j2\pi k} = -1 \cdot 1 = -1$,

$$= \frac{A}{-2\pi(1-4k^2)}[(-1-1)(1+2k) + (-1-1)(1-2k)]$$

$$= \frac{A}{-2\pi(1-4k^2)}[-4]$$
$$= \frac{2A}{\pi(1-4k^2)}$$

$$b_k = \frac{2(-1)^k}{\pi(1-4k^2)}.$$

(c) The DC component of a signal is the zeroth FS coefficient – the average of the signal over a single period.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \cos t dt = 0$$
$$b_0 = \frac{1}{\pi}\int_T |\cos t| dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t dt$$
$$= \frac{1}{\pi}[\sin t]_{-\pi/2}^{\pi/2} = \frac{1}{\pi}[1 - -1]$$
$$= \frac{2}{\pi}$$