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EE360: Signals and System I

Chapter 9
Laplace Transform

http://www.ee.unlv.edu/~b1morris/ee360/

Transform Analysis Reminder

- Goal: Represent signals as a linear combination of basic signals with two properties
  - Simple response: easy to characterize LTI system response to basic signal
  - Representation power: the set of basic signals can be use to construct a broad/useful class of signals

Fourier Series Analysis

- Simple response: Complex exponentials are eigenfunctions of LTI systems
- Representation power: Periodic signals represented as linear combination of harmonically related complex exponentials

- Limitations
  - Class of periodic signals
  - Restricted to stable systems

- Laplace Transform is a generalization

LTI Eigensignals

- Recall:
- $x(t) = e^{st} \rightarrow y(t) = H(s)e^{st}$
    - Derived from convolution integral directly
- Transfer function: $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$

- Let $s = j\omega$ for Fourier Transform
- $H(s)|_{s=j\omega} = H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$
    - We will spend more time on this in EE361 (Ch4)
    - Key it to think of filters from Ch3 since frequency has a real physical meaning

Laplace Transform

- Define LP for $s = \sigma + j\omega \in \mathbb{C}$
- $X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$

- Notation:
- $X(s) = \mathcal{L}\{x(t)\}$
- $x(t) \leftrightarrow \mathrm{X(s)}$

Important LT Pairs

- Will consider two families of signals that are useful for LTI system analysis
    - LTI systems defined by linear, constant-coefficient differential equations

- Right-sided exponential
    - $x(t) = e^{-at}u(t)$
- Left-sided exponential
    - $x(t) = -e^{-at}u(-t)$

Right-Sided Exponential I

- $x(t) = e^{-at}u(t)$

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt \\ &= -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^{\infty} = \frac{1}{s+a}[1-0] \\ &= \frac{1}{s+a} \end{aligned}$$

- $\Rightarrow e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, Re\{s\} > -a$

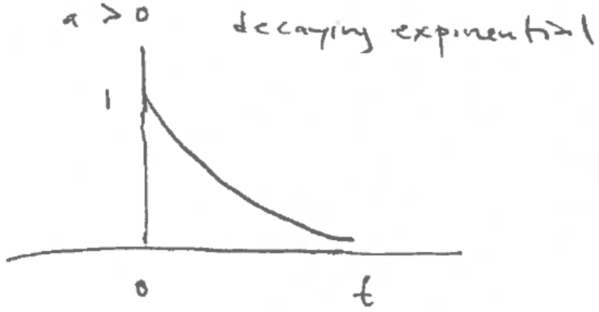
- However, consider $s = \sigma + j\omega$

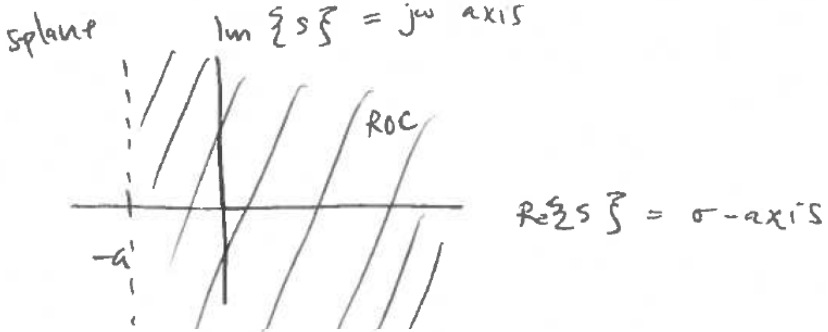
$$\begin{aligned} X(s) = X(\sigma + j\omega) &= \int_{0}^{\infty} e^{-(\sigma+j\omega+a)t}dt \\ &= \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt \\ &= \mathcal{F}\left\{e^{-(\sigma+a)t}u(t)\right\} = \frac{1}{(\sigma+a)+j\omega} \\ & \quad \sigma + a > 0 \quad \text{from Table 4.2} \\ &= \frac{1}{(\sigma+a)+j\omega} = \underbrace{\frac{1}{s+a}}_{\text{algebraic expression}} \end{aligned}$$

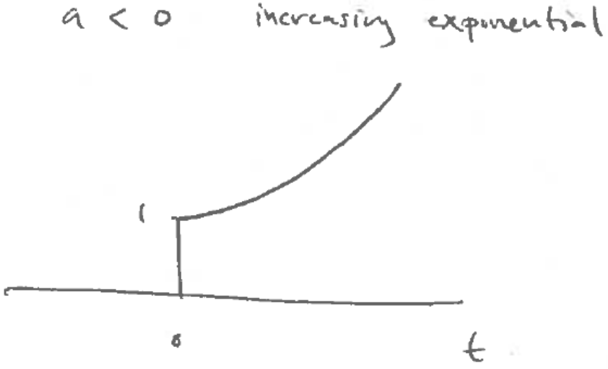
$$Re\{s\} + a > 0$$

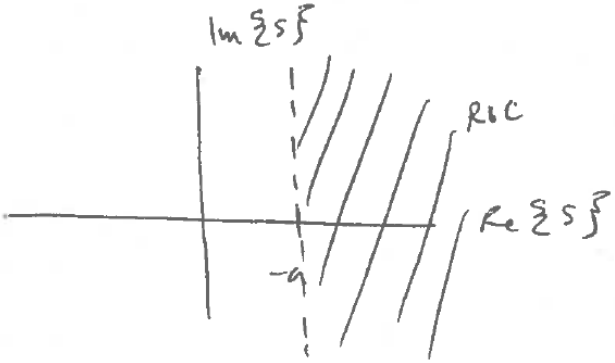
$$\underbrace{Re\{s\} > -a}_{\text{region of convergence (ROC)}}$$

Right-Sided Exponential II









- Remarks
  - For $a > 0$, FT $X(j\omega)$ exists – $X(s)$ evaluated along the $s = j\omega$ axis is within ROC
  - For $a < 0$, FT does not exist – $s = j\omega$ not in ROC
  - Right-sided signals have ROC that are right half planes

Left-Sided Exponential I

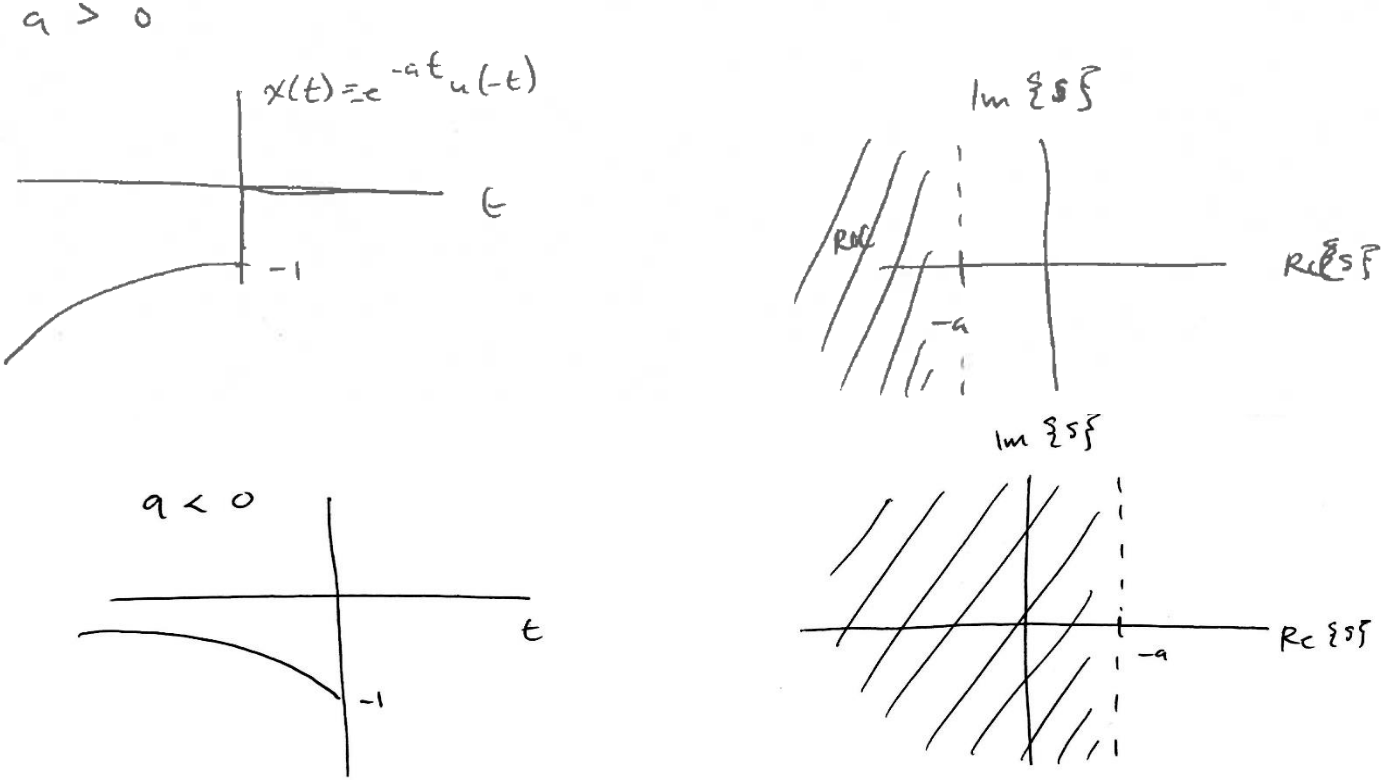
- $x(t) = -e^{-at}u(-t)$

$$
\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt = \int_{-\infty}^{0} -e^{-(s+a)t}dt \\
&= -\int_{-\infty}^{0} e^{-\sigma+a}te^{-j\omega t}dt = -\int_{-\infty}^{0} e^{-(a+\sigma+j\omega)t}dt \\
&= -\frac{1}{a+\sigma+j\omega}\left[e^{-(a+\sigma+j\omega}\right]_{-\infty}^{0} \\
&= \frac{1}{a+\sigma+j\omega} = \underbrace{\frac{1}{s+a}}_{\text{same as in previous example}}
\end{aligned}
$$

$$
\begin{aligned}
&a+\sigma < 0 \\
&Re\{s\} < -a
\end{aligned}
$$

- $\Rightarrow -e^{-at}u(-t) \leftrightarrow \frac{1}{s+a}, Re\{x\} < -a$

Left-Sided Exponential II



- Remarks
    - For $a > 0$, FT does not exist – $s = j\omega$ not in ROC
    - For $a < 0$, FT $X(j\omega)$ exists – $s = j\omega$ axis is within ROC
    - Left-sided signals have ROC that are left half planes

Example LT 1

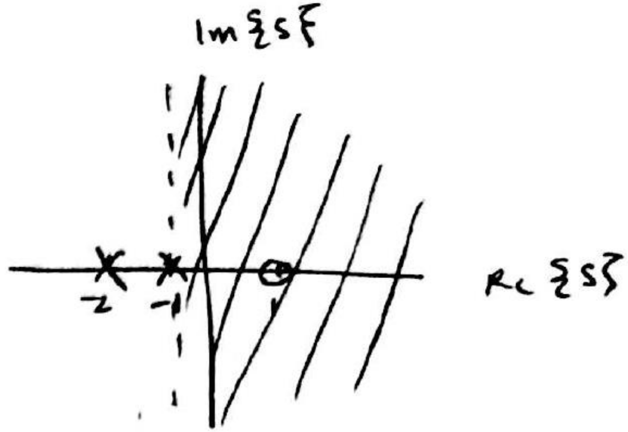
find Laplace transform

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

$$X(s) = 3\mathcal{L}\{e^{-2t}u(t)\} - 2\mathcal{L}\{e^{-t}u(t)\}$$

$$= \frac{3}{s+2} - \frac{2}{s+1}$$

$\mathrm{Re}\{s\} > -2$ $\qquad$ $\mathrm{Re}\{s\} > -1$



$$= \frac{3(s+1) - 2(s+2)}{(s+2)(s+1)} = \frac{s-1}{(s+2)(s+1)}$$

o - zeros $\Rightarrow X(s) \overset{=0}{\text{@ zero}}$

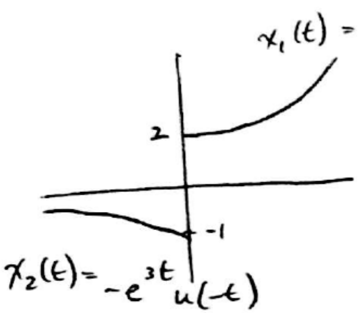
X - poles $\Rightarrow X(s) = \infty$ @ pole

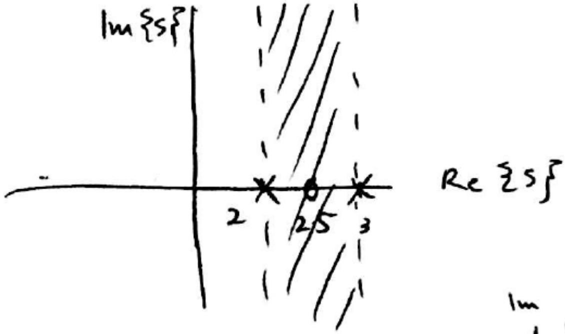
RoC: where all $\overset{\text{individual terms}}{\wedge}$ terms converge $\Rightarrow$ intersection

$\mathrm{Re}\{s\} > -1$

Example LT 2

$x(t) = e^{2t}u(t) + e^{3t}u(-t)$

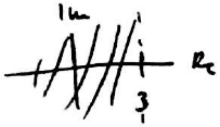




⇒ two sided signal, ROC in vertical strips.

$x_1(t) = e^{2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-2}$ $\quad Re\{s\} > 2$

$x_2(t) = -e^{3t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-3}$ $\quad Re\{s\} < 3$



$x(t) = x_1(t) + x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s) + X_2(s)$

$$\frac{1}{s-2} + \frac{1}{s-3} = \frac{2s-5}{(s-2)(s-3)}$$

$ROC = ROC_1 \cap ROC_2$

$+2 < Re\{s\} < 3$

ROC cannot contain any poles

$\Rightarrow X(2) = X(3) = \infty$

no convergence.

Laplace Transform Tables

- Previously, we computed the LT directly from the integral definition
  - Learned that there is an algebraic form for LT as well as the region of convergence (ROC)
  - Both are needed to uniquely identify the LT

- More often, we will use tables of known transform pairs (Table 9.2) and transform properties (Table 9.1).

ROC for Laplace Transform

- Algebraic expressions of LT are not sufficient for distinguishing LT, must also include ROC

- 8 properties of ROC
- 1: ROC consists of strips parallel to $j\omega$-axis
  (right-half plane, left-half plane, strip)
  ROC only depends on $Re\{s\} = \sigma \Rightarrow$ vertical lines

- 2: Rational $X(s)$ does not contain any poles
  $X(s)$ is infinite at pole ⇒ not stable

ROC for Laplace Transform II

- 3: If $x(t)$ is finite duration and absolutely integrable, then ROC is the entire s-place

- 4: If $x(t)$ is right-sided and $\{s|Re\{s\} = \sigma_0\} \subseteq$ ROC, then $\{s|Re\{s\} > \sigma_0\} \subseteq$ ROC
  - Right-sided ROC

- 5: If $x(t)$ is left-sided and $\{s|Re\{s\} = \sigma_0\} \subseteq$ ROC, then $\{s|Re\{s\} < \sigma_0\} \subseteq$ ROC
  - Left-sided ROC

ROC for Laplace Transform III

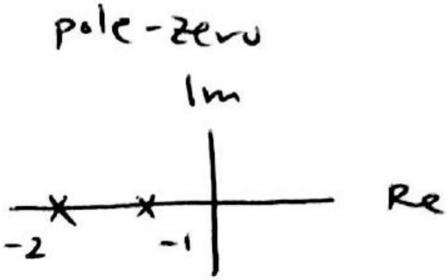
- 6: If $x(t)$ is two-sided (not-bounded) and $\{s|Re\{s\} = \sigma_0\} \subseteq$ ROC, then the ROC is a vertical strip containing the line $Re\{s\} = \sigma_0$

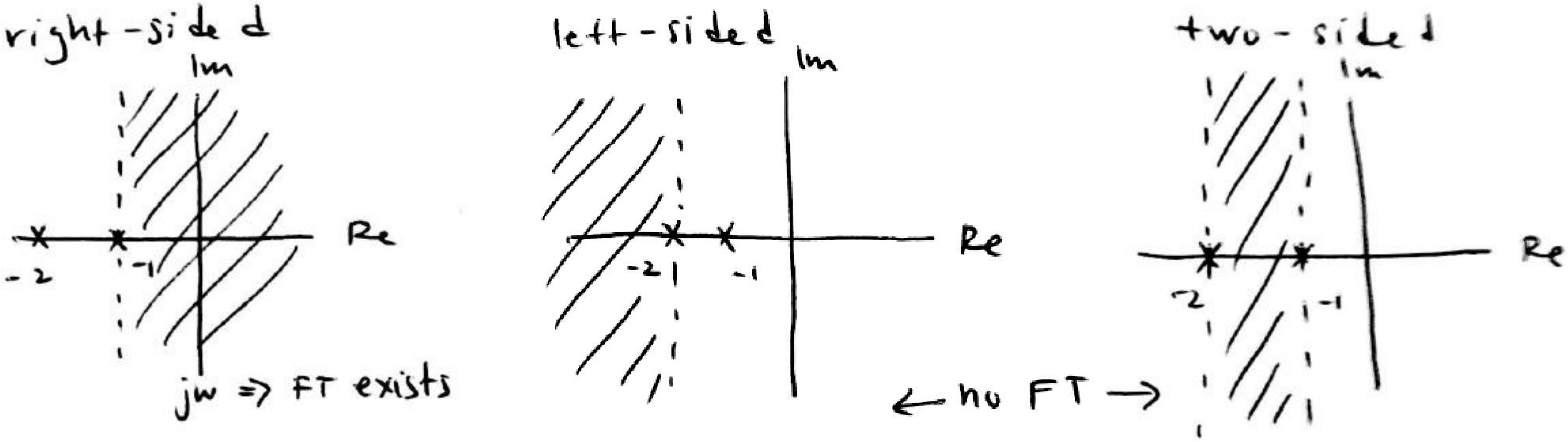
- 7: If the LT $X(s)$ is rational, the ROC is bounded by poles or extends to infinity (pole @ ∞)

- 8: If $X(s)$ is rational, then
  - $x(t)$ right-sided, ROC is the right-half plane to the right of the right-most pole
  - $x(t)$ left-sided, ROC is the left-half plane to the left of the left-most pole

Example ROC Properties

- $X(s) = \dfrac{1}{(s+1)(s+2)}$





Inverse LT

- By using inverse FT we can find the definition of the inverse LT

$$x(t) = \frac{1}{2\pi} \int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds$$

  - The synthesis equation is a weighted sum of complex exponentials
- Note: this is a contour integral along any vertical line in the ROC of s-plane!
  - We won’t do it this way, instead will rely on PFE techniques in lookup tables

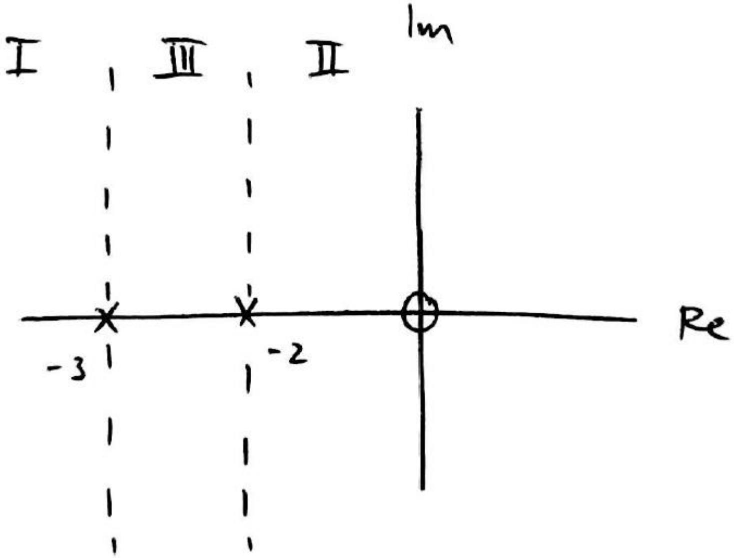
Example Inverse LT

$$X(s) = \frac{s}{s^2 + 5s + 6}$$
$$= \frac{s}{(s+2)(s+3)}$$
$$= \frac{A}{s+2} + \frac{B}{s+3}$$

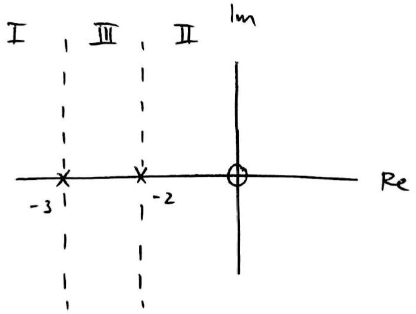
- Solve for partial terms
- $A = \left.\frac{s}{s+3}\right|_{s=-2} = -2$
- $B = \left.\frac{s}{s+2}\right|_{s=-3} = 3$
- Combine for final answer

$$X(s) = -\frac{2}{s+2} + \frac{3}{s+3}$$

- What are the possible ROC's?



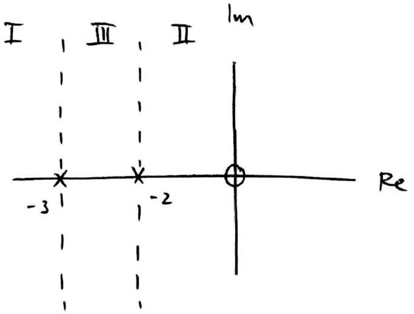
Example Inverse LT II



- ROC I
  - $Re\{s\} < -3 \quad \Rightarrow$ left-sided signal
  - $X(s) = -\frac{2}{s+2} + \frac{3}{s+3} \leftrightarrow x(t) =$ $-\ 2[-e^{-2t}u(-t)] + 3[-e^{-3t}u(-t)]$
  - Notice: the use of left-sided transform pair

- ROC II
  - $Re\{s\} > -2 \quad \Rightarrow$ right-sided signal
  - $X(s) \leftrightarrow x(t) = -2[e^{-2t}u(t)] + 3[e^{-3t}u(t)]$

Example Inverse LT III



- ROC III
  - $-3 < Re\{s\} < -2 \implies$ two-sided signal
  - Want intersection of
    - Left-sided $\frac{1}{s+2}$ term $(Re\{s\} < -2)$
    - Right-sided $\frac{1}{s+3}$ term $(Re\{s\} > -3)$
  - $X(s) \leftrightarrow x(t) = -2[\underset{\uparrow\atop\text{Left-sided}}{-e^{-2t}}u(-t)] + 3[\underset{\uparrow\atop\text{Right-sided}}{e^{-3t}}u(t)]$

Properties of LT (Table 9.1)

- Linearity
    - $x_1(t) \leftrightarrow X_1(s), \mathrm{ROC} = R_1$
    - $x_2(t) \leftrightarrow X_2(s), \mathrm{ROC} = R_2$
  - $ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)$
    - $\mathrm{ROC} \supset R_1 \cap R_2$
    - Contains at the very least intersection but could be more

Example: Linearity

$$x(t) = x_1(t) - x_2(t)$$

$$X_1(s) = \frac{1}{s+1}$$

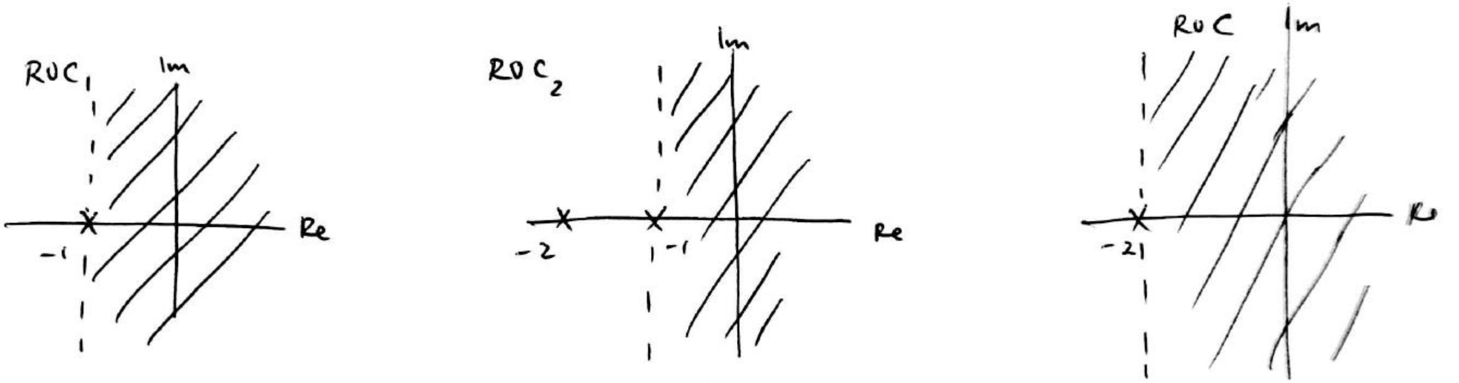
$\text{ROC}_1 \ \text{Re}\{s\} > -1$

$$X_2(s) = \frac{1}{(s+1)(s+2)}$$

$\text{ROC}_2 \ \text{Re}\{s\} > -1$

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+2)(s+1)}$$

$$= \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$ due to pole-zero cancellation



Note: ROC larger than $\text{ROC}_1 \cap \text{ROC}_2$ because zero and pole cancel @ $s = -1$

Properties of LT II

- Time-shift
    - $x(t) \leftrightarrow X(s), \mathrm{ROC} = R$
    - $x(t-t_0) \leftrightarrow e^{-st_0}X(s)$, ROC $= R$
- S-domain shift
    - $e^{s_0t}x(t) \leftrightarrow X(s-s_0)$, ROC $= R + Re\{s_0\}$

- Convolution property
    - $x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s)$, ROC $\supset \mathrm{R}_1 \cap R_2$
    - Key for LTI systems described by differential equations

Properties of LT III

- Differentiation in time
  - $\frac{dx(t)}{dt} \leftrightarrow sX(s)$, ROC $\supset R$
  - More generally,
  - $\frac{d^k}{dt^k} x(t) \leftrightarrow s^k X(s)$, ROC $\supset R$
- Integration in time
  - $\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s} X(s)$, ROC $\supset R \cap \{s|Re\{s\} > 0\}$
    - With $x(t) = \delta(t)$,
    - $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \leftrightarrow \frac{1}{s}$, ROC $= \{s|Re\{s\} > 0\}$
  - Integration is useful for block diagram representations

LTI Systems and LT

- Reminder,

- Convolution
- $y(t) = x(t) * h(t) \longleftrightarrow Y(s) = X(s)H(s)$
  - $H(s)$ is the system function or transfer function

- Eigensignal
- $x(t) = e^{st} \longleftrightarrow y(t) = H(s)e^{st}$
  - For s in the ROC of $H(s)$
  - If $s = j\omega$ is in the ROC, then the FT exists and $H(j\omega)$ is the frequency response

LTI System Properties from $H(s)$

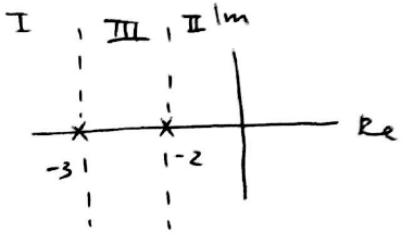
- Many properties of LTI system can be determined directly from system function $H(s)$
  - 1: Causality – a causal system has an ROC that is a right half plane
  - 2: Stability – a system is stable iff ROC of $H(s)$ contains the entire $j\omega$-axis
- For rational $H(s)$ [ratio of polynomial functions as in diff eqns], more can be specified
  - 1r: Causality – the system is causal iff ROC is the right-half plane to the right of the right-most pole
  - 2r: Stability – a causal system is stable iff all poles lie in the half-plane to the left of the $j\omega$-axis
    - All poles have negative real parts

Example

Example

$$H(s) = \frac{s}{(s+2)(s+3)} = \frac{-2}{s+2} + \frac{3}{s+3}$$

↑
rational



ROC I

not causal (anti-causal, not right-half plane) not stable (no jw-axis)

$h_I(t) = 2e^{-2t}u(-t) - 3e^{-3t}u(-t)$

ROC II

causal, right-half plane, stable, contains jw-axis,

$h_{II}(t) = -2e^{-2t}u(t) + 3e^{-3t}u(t)$

ROC III

not causal, not stable,

$h_{III}(t) = 2e^{-2t}u(-t) + 3e^{-3t}u(t)$

Differential Equation LTI Systems

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- Use differentiation in time-domain property

$$\left[\sum_{K=0}^{N} a_k s^k\right] Y(s) = \left[\sum_{K=0}^{M} b_k s^k\right] X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

← Zeros

← Poles

Example

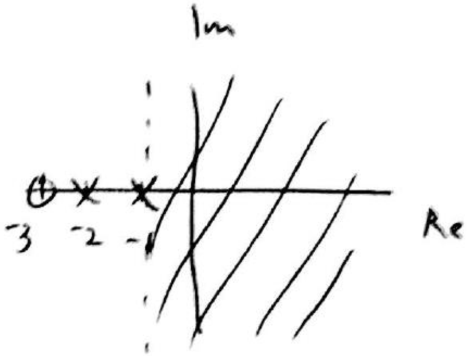
given causal LTI system described by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$$

find h(t)

$$Y(s)\left[s^2 + 3s + 2\right] = X(s)\left[s + 3\right]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+2)(s+1)} \Rightarrow$$



$$= \frac{A}{s+2} + \frac{B}{s+1} = \frac{-1}{s+2} + \frac{2}{s+1}$$

causality implies

$$ROC = \{s \mid Re\{s\} > -1\}$$

$$A = \left.\frac{s+3}{s+1}\right|_{s=-2} = \frac{1}{-1} = -1$$

$$B = \left.\frac{s+3}{s+2}\right|_{s=-1} = \frac{2}{1} = 2$$

$$h(t) = -e^{-2t}u(t) + 2e^{-t}u(t)$$

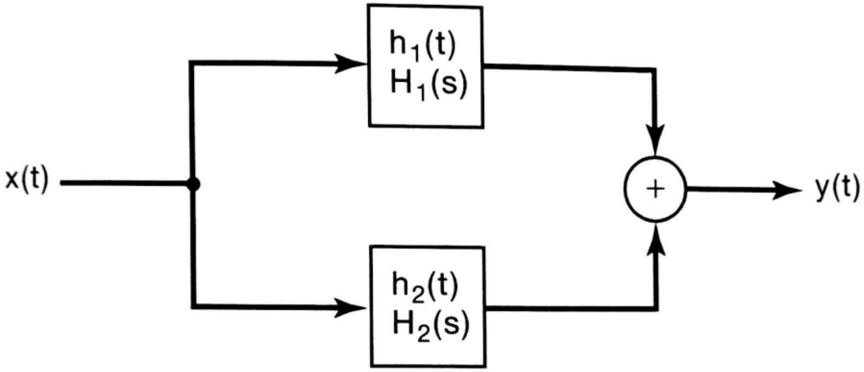
System Function Algebra/Block Diagrams

- System Functions for:
  - Parallel interconnections
  - Series interconnections
  - Feedback interconnections

- Block Diagrams for LTI Systems
  - Basic building blocks
  - Direct Forms
  - Cascade Form
  - Parallel Form

Parallel Interconnections

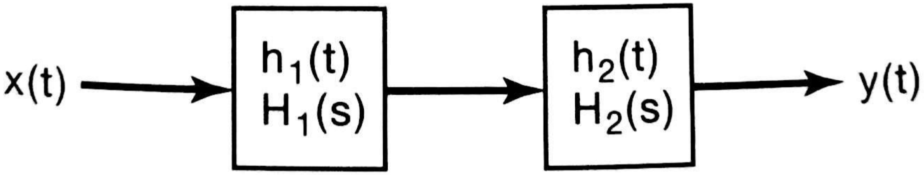
- $h(t) = h_1(t) + h_2(t) \leftrightarrow H(s) = H_1(s) + H_2(s)$



- $y(t) = x(t) * h_1(t) + x(t) * h_2(t)$
- $y(t) = x(t) * \underbrace{[h_1(t) + h_2(t)]}_{H(s) = H_1(s) + H_2(s)}$

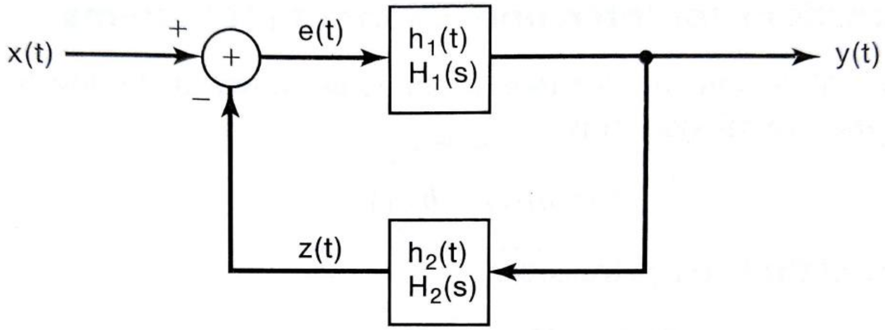
Series Interconnections

- $h(t) = h_1(t) * h_2(t) \leftrightarrow H(s) = H_1(s)H_2(s)$



- Define intermediate signal
  - $z(t) = x(t) * h_1(t)$
- $y(t) = z(t) * h_2(t) = [x(t) * h_1(t)] * h_2(t)$
- $y(t) = x(t) * \underbrace{[h_1(t) * h_2(t)]}_{h(t) \leftrightarrow H(s) = H_1(s)H_2(s)}$

Feedback Interconnection



$$\left.\begin{aligned} y(t) &= e(t) * h_1(t) \\ z(t) &= y(t) * h_2(t) \\ e(t) &= x(t) - z(t) \end{aligned}\right\}$$

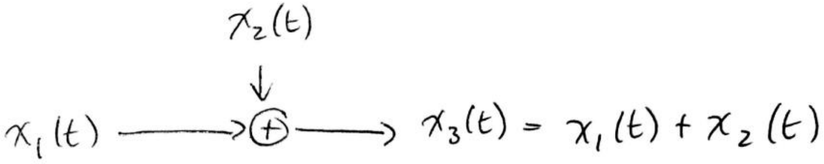
$$\begin{aligned} y(t) &= [x(t) - z(t)] * h_1(t) \\ &= [x(t) - y(t) * h_2(t)] * h_1(t) \\ &= x(t) * h_1(t) + y(t) * h_2(t) * h_1(t) \\ Y(s) &= X(s)H_1(s) + Y(s)H_2(s)H_1(s) \end{aligned}$$

$$\begin{aligned} \Rightarrow Y(s) + Y(s)H_1(s)H_2(s) &= X(s)H_1(s) \\ Y(s)\left[1 + H_1(s)H_2(s)\right] &= X(s)H_1(s) \end{aligned}$$

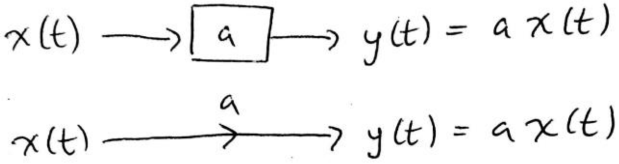
$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Block Diagram Building Blocks

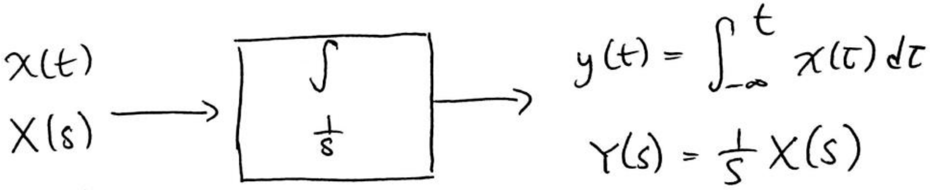
- Addition



- Scaling



- Integration



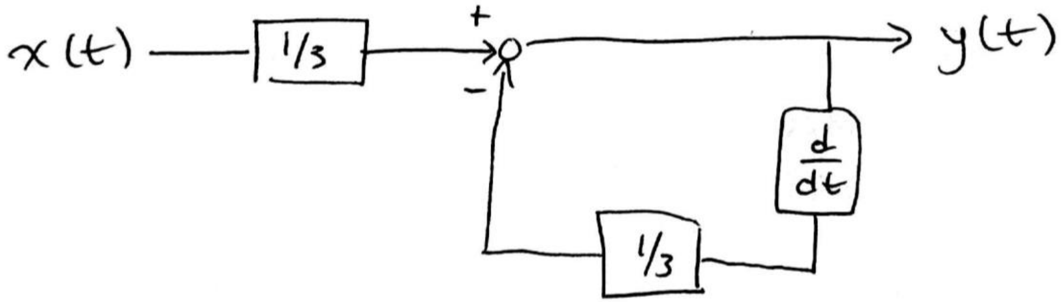
- Notice integration is used rather than differentiation despite LTI systems being given in diff eq form since integration is easier to implement than derivatives
  - E.g. integration is RC circuit

Block Diagram Forms

- Direct Forms
    - Discussed earlier in class
- Cascade Form
    - Series interconnection from factorization of system function
- Parallel Form
    - Parallel interconnection from PFE of system function
    - Polynomial long division may be required

Example CT LTI Block Diagrams I

- Given $H(s) = \frac{1}{s+3}$, draw block diagram
  - $H(s) = \frac{Y(s)}{X(s)} \Rightarrow X(s) = (s+3)Y(s)$
  - $\Rightarrow x(t) = \frac{d}{dt}y(t) + 3y(t)$
- $y(t) = \frac{1}{3}x(t) - \frac{1}{3}\frac{d}{dt}y(t)$



derivative form

⇒ will be most similar to DT time

Example CT LTI Block Diagrams II

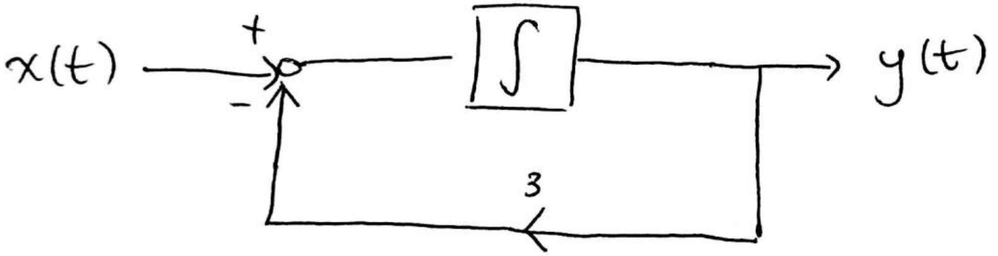
- $x(t) = \frac{d}{dt}y(t) + 3y(t)$
- $\Rightarrow \frac{d}{dt}y(t) = x(t) - 3y(t)$

- $y(t) = \int_{-\infty}^{t}[x(\tau) - 3y(\tau)]d\tau$

$$\frac{d}{dt}y(t) = x(t) - 3y(t)$$

②

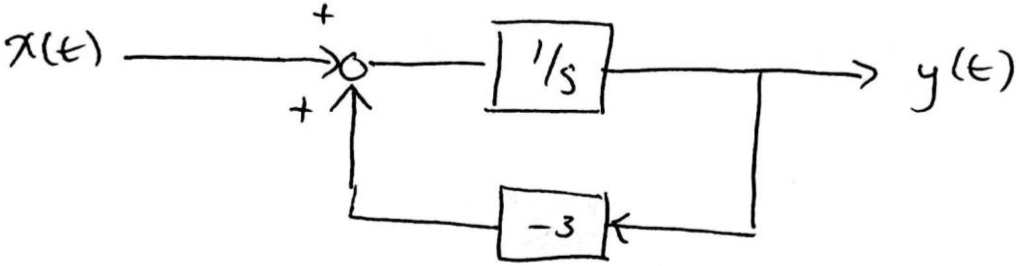
$$y(t) = \int_{-\infty}^{t}\left[x(\tau) - 3y(\tau)\right]d\tau$$



integral in time domain

Example CT LTI Block Diagrams III

- $H(s) = \frac{1}{s+3} = \frac{1/s}{1+3/s}$
- $H(s) = \frac{1/s}{1-(-3)(1/s)} = \frac{H_1(s)}{1-H_2(s)H_1(s)}$
  - $H_1(s) = 1/s,\ H_2(s) = -3$



same as ② but 1/s replaces $\boxed{\int}$ block.

- Notice: for this technique you want to divide by the largest order of s in $H(s)$ to have the total number of needed integrals

Unilateral Laplace Transform

- Previously, bilateral LT

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt, \text{ with ROC} \qquad x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

- In contrast, unilateral LT

$$X_u(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt \qquad x(t) \overset{U\mathcal{L}}{\longleftrightarrow} X_u(s)$$

  - Integration is from $[0^-, \infty]$ rather than $[-\infty, \infty]$
  - $0^- \Rightarrow \lim_{\epsilon \to 0} 0 - \epsilon$
    - Allows for values of input at $t = 0$ (e.g. $\delta(t)$)

Unilateral LT Notes

- Results in unique transform pairs (causal)
  - No need for specifying ROC → always right-sided
- However, signals that are the same for $t \geq 0$ (independent of values for $t < 0$) are the same
  - $U\mathcal{L}\{x(t)\} = \mathcal{L}\{x(t)u(t)\}$

- We care about ULT because it is useful for study of causal LTI diff eq systems with non-zero initial conditions (not initially at rest)
  - Finally learning a better way of solving diff eq than the particular and homogenous solutions

ULT Properties

- Luckily, these are basically the same as bilateral LT
- Time Differentiation Property
- $\frac{d}{dt}x(t) \leftrightarrow sX_u(s) - x(0^-)$
  - Incorporates initial condition in extra term
  - Derived using ULT integral with integration by parts ($dv = \frac{d}{dt}x(t)dt$ and $u = e^{-st}$)
- $\frac{d^2}{dt^2}x(t) \leftrightarrow s^2X_u(s) - sx(0^-) - \frac{d}{dt}x(0^-)$
  - $\frac{d}{dt}x(0^-) = x'(0^-)$
- $\frac{d^n}{dt^n}x(t) \leftrightarrow s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$

Example: ULT I

Find the response to LTI system described by

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + x(t)$$

to input $x(t) = e^{-4t}u(t)$ with initial conditions $y(0^-) = 2$ and $y'(0^-) = 1$.

$$x(t) = e^{-4t}u(t) \longleftrightarrow X_u(s) = \frac{1}{s+4}$$

$$\frac{d}{dt}x(t) \longleftrightarrow sX_u(s) - x(0^-) = sX_u(s) - 0 = \frac{s}{s+4}$$

From the diff eq. and taking ULT

$$s^2Y_u(s) - 2s - 1 + 5(sY_u(s) - 2) + 6Y_u(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

$$Y_u(s)\left[s^2 + 5s + 6\right] + \left[-2s - 11\right] = \frac{s+1}{s+4}$$

$$Y_u(s)\left[s^2 + 5s + 6\right] = \underbrace{\left[2s + 11\right]}_{\substack{\text{initial}\\\text{conditions}}} + \underbrace{\frac{s+1}{s+4}}_{\substack{\text{input terms}\\\text{from } x(t)}}$$

$$Y_u(s) = \underbrace{\left[\frac{2s+11}{s^2+5s+6}\right]}_{\substack{\text{zero-input component}\\\text{(initial conditions)}}} + \underbrace{\left[\frac{s+1}{(s+4)(s^2+5s+6)}\right]}_{\substack{\text{zero-state component}\\\text{(input terms)}}}$$

Example: ULT II

$$Y_u(s) = \underbrace{\left[\frac{2s+11}{s^2+5s+6}\right]}_{\substack{\text{zero-input component} \\ \text{(initial conditions)}}} + \underbrace{\left[\frac{s+1}{(s+4)(s^2+5s+6)}\right]}_{\substack{\text{zero-state component} \\ \text{(input terms)}}}$$

$$Y(s) = \underbrace{\left[\frac{A}{s+3} + \frac{B}{s+2}\right]}_{ZIR} + \underbrace{\left[\frac{C}{s+4} + \frac{D}{s+3} + \frac{E}{s+2}\right]}_{ZSR \ \circledast}$$

$$A = \left.\frac{2s+11}{s+2}\right|_{s=-3} = \frac{5}{-1} = -5$$

$$B = \left.\frac{2s+11}{s+3}\right|_{s=-2} = \frac{7}{1} = 7$$

$$C = \left.\frac{s+1}{(s+3)(s+2)}\right|_{s=-4} = \frac{-3}{(-1)(-2)} = -3/2$$

$$D = \left.\frac{s+1}{(s+4)(s+2)}\right|_{s=-3} = \frac{-2}{(1)(-1)} = 2$$

$$E = \left.\frac{s+1}{(s+4)(s+3)}\right|_{s=-2} = \frac{-1}{(-2)(-1)} = -1/2$$

$$Y(s) = \left[\underbrace{\frac{-5}{s+3} + \frac{7}{s+2}}_{ZIC}\right] + \left[\underbrace{\frac{-3/2}{s+4} + \frac{2}{s+3} + \frac{-1/2}{s+2}}_{ZSC}\right]$$

$$y(t) = \underbrace{-5e^{-3t}u(t) + 7e^{-2t}u(t)}_{\text{zero-input response}} + \underbrace{-3/2\, e^{-4t}u(t) + 2e^{-3t}u(t) - 1/2\, e^{-2t}u(t)}_{\text{zero-state response}}$$

response when input = 0

⇒ response to initial conditions

changes with i.c.

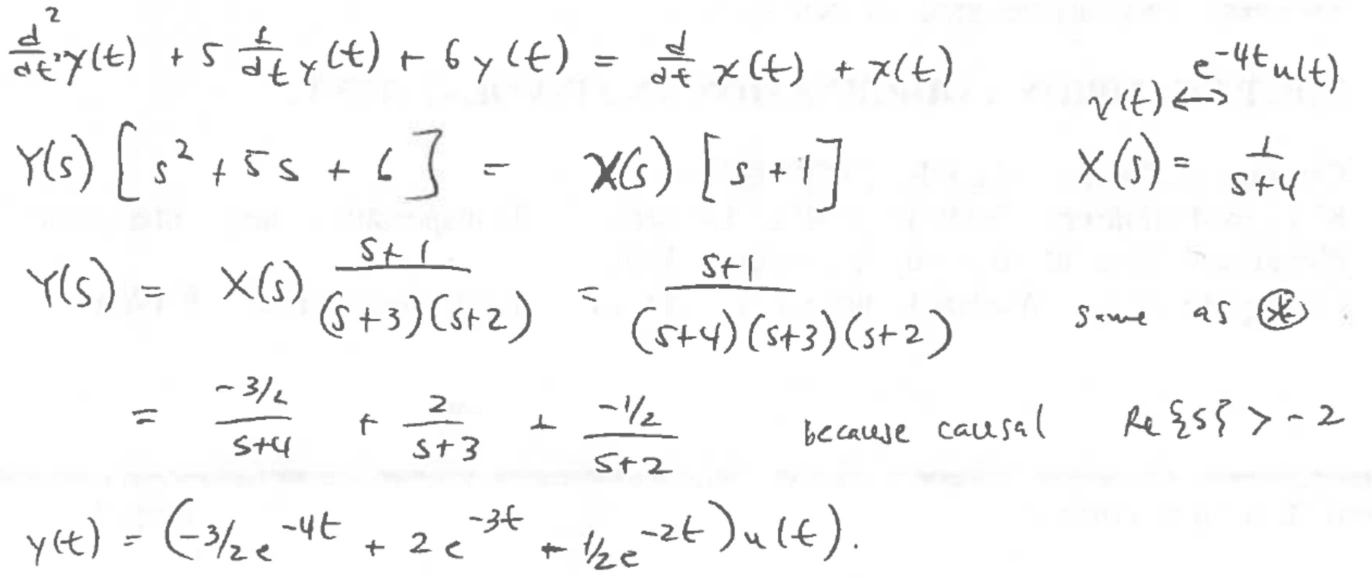
response when initial conditions = 0 (initial rest)

does not change with i.c.

⇒ this is our bilateral response using only causal pairs.

Example: ULT III

- Repeat problem with bilateral LT



$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + x(t)$$

$x(t) \leftrightarrow e^{-4t}u(t)$

$$Y(s)\left[s^2 + 5s + 6\right] = X(s)\left[s+1\right]$$

$X(s) = \frac{1}{s+4}$

$$Y(s) = X(s)\frac{s+1}{(s+3)(s+2)} = \frac{s+1}{(s+4)(s+3)(s+2)}$$

Same as $\circledast$

$$= \frac{-3/2}{s+4} + \frac{2}{s+3} + \frac{-1/2}{s+2}$$

because causal $Re\{s\} > -2$

$$y(t) = \left(-3/2\,e^{-4t} + 2e^{-3t} + 1/2\,e^{-2t}\right)u(t).$$

- Note: in this case same as not having initial conditions