EE360: SIGNALS AND SYSTEMS I

CH2: LINEAR TIME-INVARIANT SYSTEMS

INTRODUCTION

CHAPTER 2.0

LTI SYSTEMS

- Important class of systems because many real physical processes have these properties

- LTI systems have properties that have been studied extensively leading to powerful and effective theory for analyzing their behavior

DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

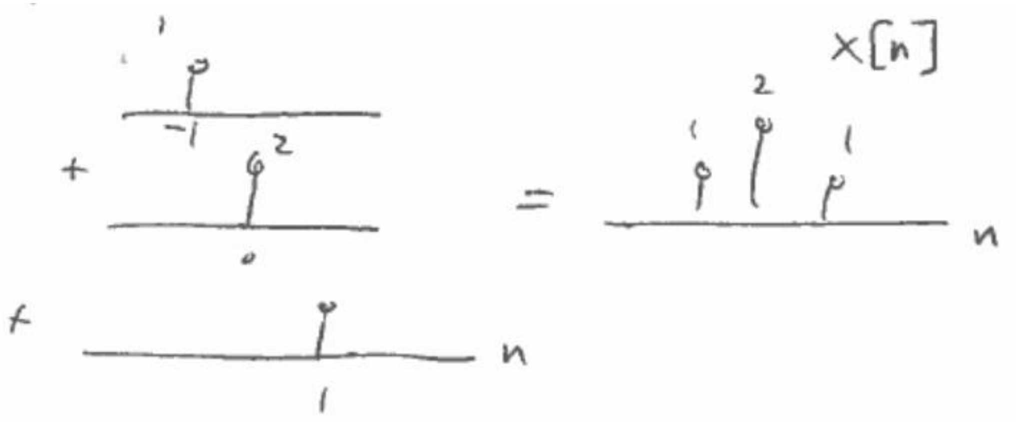
CHAPTER 2.1

REMINDER: REPRESENTATION

- A signal can be composed of scaled and shifted impulses
  - Remember sifting and representation properties

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Scale factor   Shifted impulse



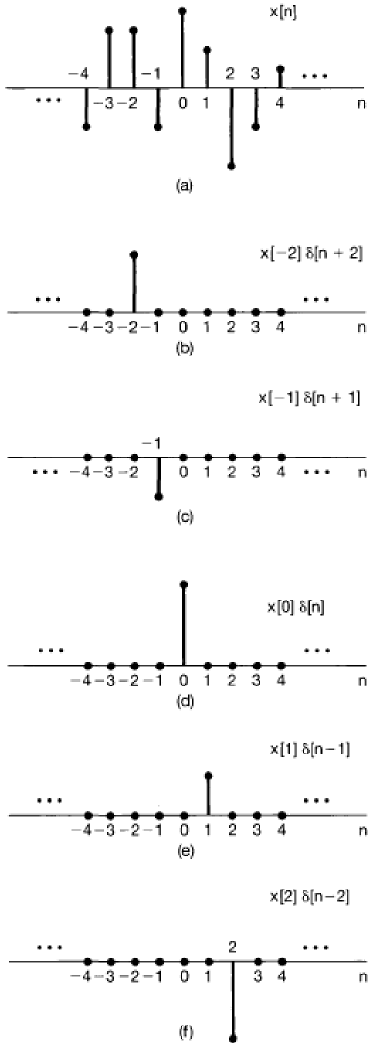
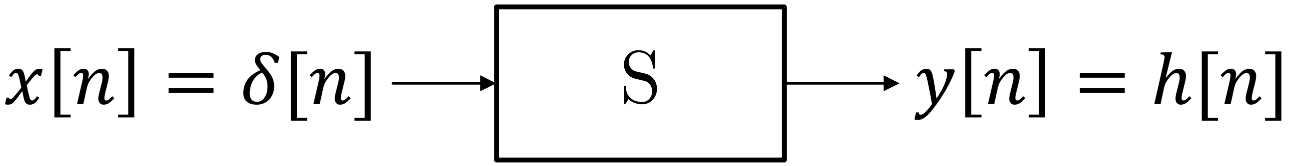


Figure 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

IMPULSE RESPONSE $h[n]$

- Response of LTI system to delta input



$$x[n] \longrightarrow y[n]$$
$$\delta[n] \longrightarrow h[n]$$

  - $h[n]$ – (unit) impulse response
- Time invariance of S
  - If $\delta[n] \to h[n]$, then $\delta[n-k] \to h[n-k] \quad \forall k \in \mathbb{Z}$
- Linearity of S
  - If $\delta[n] \to h[n]$, then $\sum_k a_k \delta[n-k] \to \sum_k a_k h[n-k]$
    - $a_k \in \mathbb{C}, \forall k \in K \subseteq \mathbb{Z}$

CONVOLUTION

- Using representation property
  - $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
  - $x[n] = \sum_{k=-\infty}^{\infty} a_k \delta[n-k]$
    - $a_k = x[k]$
- By LTI properties

$$x[n] = \sum_k x[k]\delta[n-k] \longrightarrow \underbrace{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{\text{convolution}}$$

- Convolution operation

$$y[n] = x[n] * h[n]$$

with $*$ the convolution operator

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= h[n] * x[n]$$

LTI SYSTEM REPRESENTATION

- Convolution formula allows the computation of system output for any input
- If the impulse response $h[n]$ is known, the LTI system is completely specified → know everything about the system

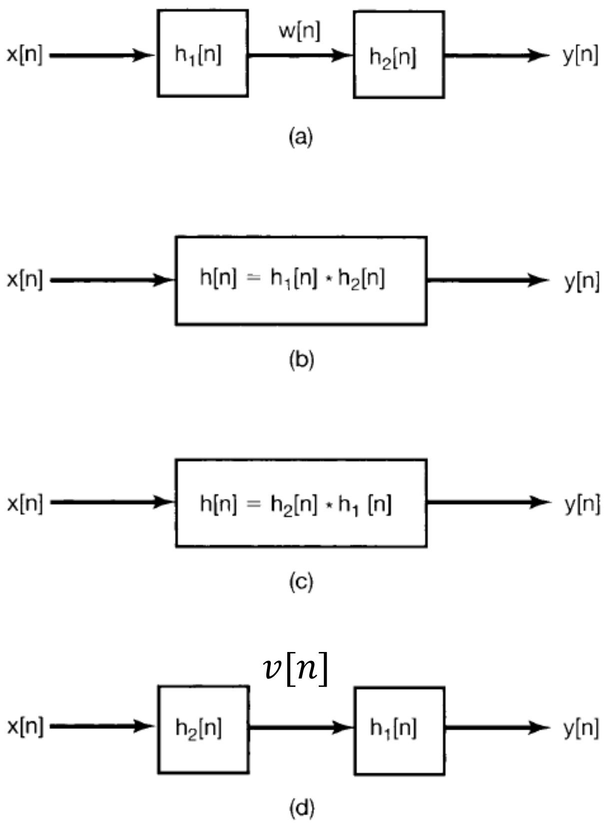
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

SOME QUICK PROPERTIES

- Commutative property
  - $x[n] * h[n] = h[n] * x[n]$
- Distributive property
  - $x[n] * (h_1[n] + h_2[n]) = (x[n] * h_1[n]) + (x[n] * h_2[n])$
- Associative property
  - $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$

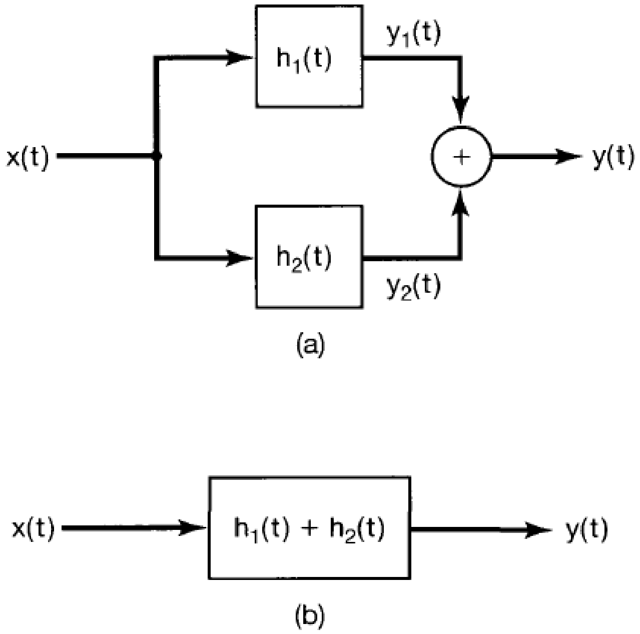
CONSEQUENCES

- Order of convolution does not matter



Differing intermediate signals

- Diagram simplification



INTERPRETATIONS OF CONVOLUTION I

- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

- ① Fix the value of k
- Define $w_k[n] = x[k]h[n-k]$
  - Function of time variable n
  - Scaled and shifted impulse response
- Output signal $y[n] = \sum_k w_k[n]$
  - Sum over all signals $w_k[n]$

INTERPRETATIONS OF CONVOLUTION II

- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

- ② Fix a value of n
- Define $v_n[k] = x[k]h[n-k]$
  - Function (signal) of time variable k
- Output signal $y[n] = \sum_k v_n[k]$
  - Sum over single signal $v_n[k]$
  - Output is built from a single value at a time

① CONVOLUTION – SCALED/SHIFTED $h[n]$

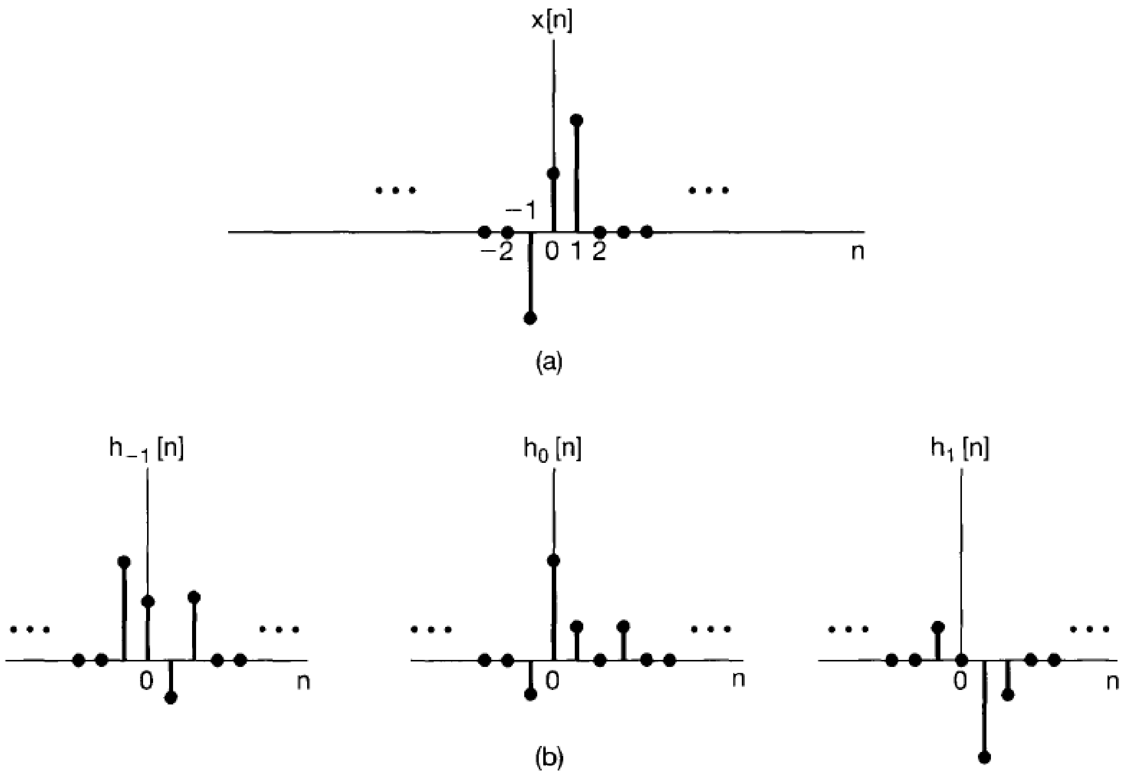
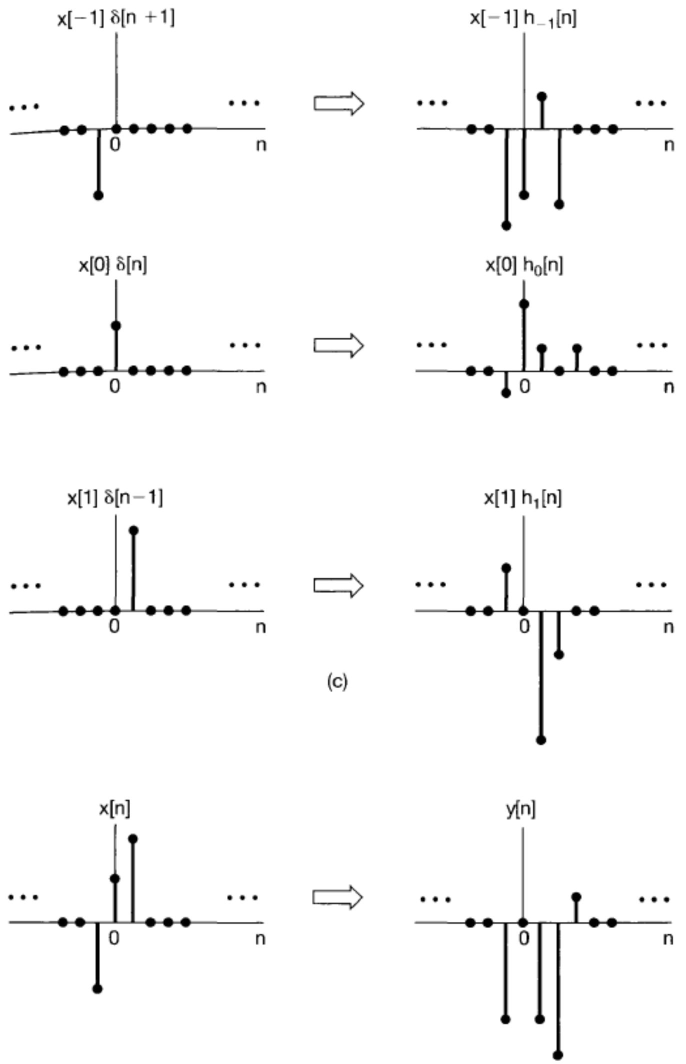
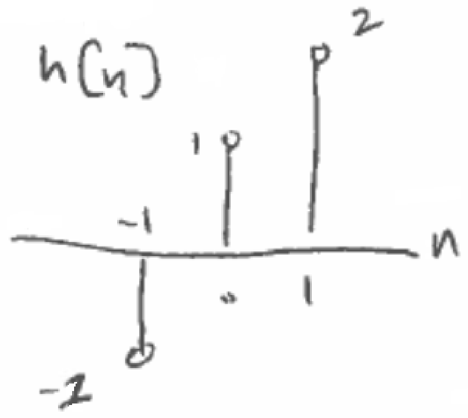


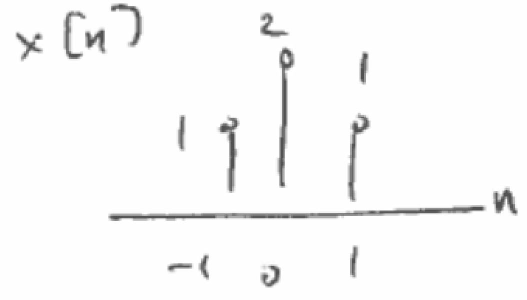
Figure 2.2 Graphical interpretation of the response of a discrete-time linear system as expressed in eq. (2.3).



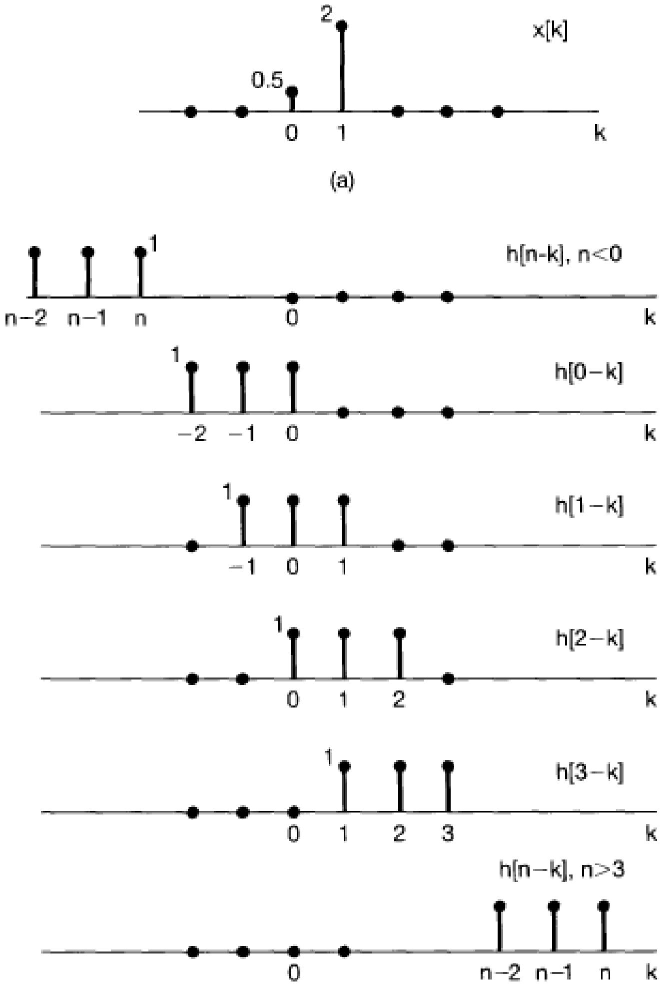
EXAMPLE: ① CONVOLUTION

- $h[n] = -\delta[n+1] + \delta[n] + 2\delta[n-1]$
- $x[n] = \delta[n+1] + 2\delta[n] + \delta[n-1]$
- Find output $y[n] = x[n] * h[n]$





② CONVOLUTION – FLIP AND DRAG

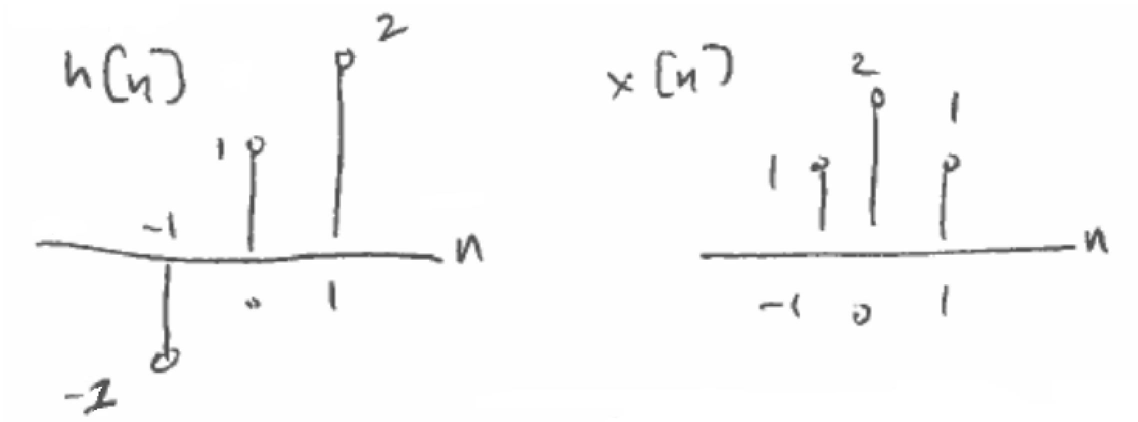


- Flip and shift $h \rightarrow h[n-k]$
  - Only consider overlap

- $y[n] = 0 \qquad n < 0$
- $y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5$
- $y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5$
- $y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5$
- $y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0 = 2.0$

EXAMPLE: ② CONVOLUTION

- $h[n] = -\delta[n+1] + \delta[n] + 2\delta[n-1]$
- $x[n] = \delta[n+1] + 2\delta[n] + \delta[n-1]$
- Find output $y[n] = x[n] * h[n]$

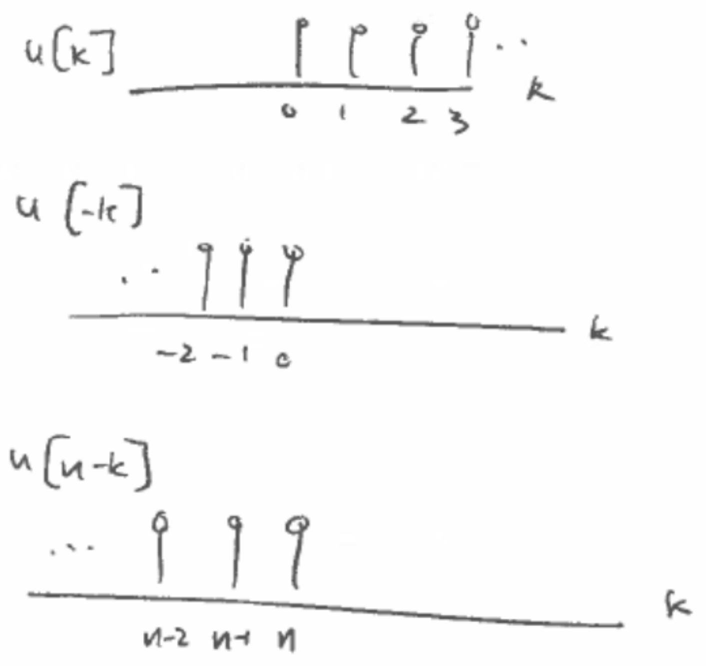


- To completely define the signal $y[n]$ signal, must give value $y[n_0]$ for all times n_0

- Steps:
  1. Choose n value
  2. Compute $v_n[k] = x[k]h[n-k]$
  3. Sum over k over $x[k]h[n-k]$ signal
  4. Slide to new time n and repeat 2-3 until all n visited

STEP RESPONSE $s[n]$

$$x[n] = u[n] \longrightarrow \boxed{h[n]} \longrightarrow s[n] = u[n] * h[n]$$



- $s[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k]$
- $s[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k]$
- $\qquad = \sum_{k=-\infty}^{n} h[k]$

- Since
  - $\delta[n] = u[n] - u[n-1]$
  - $h[n] = s[n] - s[n-1]$
- The step response (in addition to impulse response) completely determines an LTI system

USEFUL SUMMATION FORMULAS

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \frac{1-\alpha^N}{1-\alpha} & \alpha \neq 1 \\ N & \alpha = 1 \end{cases}$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

$$\sum_{n=k}^{N-1} \alpha^n = \begin{cases} \frac{\alpha^k-\alpha^N}{1-\alpha} & \alpha \neq 1 \\ N-k & \alpha = 1 \end{cases}$$

$$\sum_{n=k}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \qquad |\alpha| < 1$$

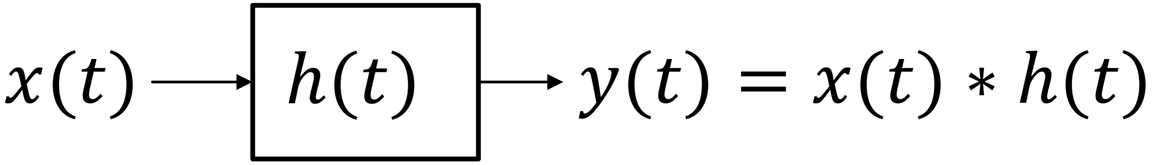
EXAMPLE: DT CONVOLUTION

- $x[n] = \alpha^n u[n]$
- Find $y[n] = x[n] * y[n]$
- $h[n] = u[n] - u[n-6]$

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

CHAPTER 2.2

CONTINUOUS TIME LTI SYSTEM



- $x(t) \longrightarrow y(t)$
- $\delta(t) \longrightarrow h(t)$

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \end{aligned}$$

See Riemann sum approximation for derivation in book

PROPERTIES OF LTI SYSTEMS

CHAPTER 2.3

QUICK PROPERTIES

- Commutative property
  - $x(t) * h(t) = h(t) * x(t)$
  - Flip signal that is most convenient

- Distributive property
  - $x(t) * \left(h_1(t) + h_2(t)\right) = \left(x(t) * h_1(t)\right) + (x(t) * h_2(t))$

- Associative property
  - $x(t) * \left(h_1(t) * h_2(t)\right) = \left(x(t) * h_1(t)\right) * h_2(t)$

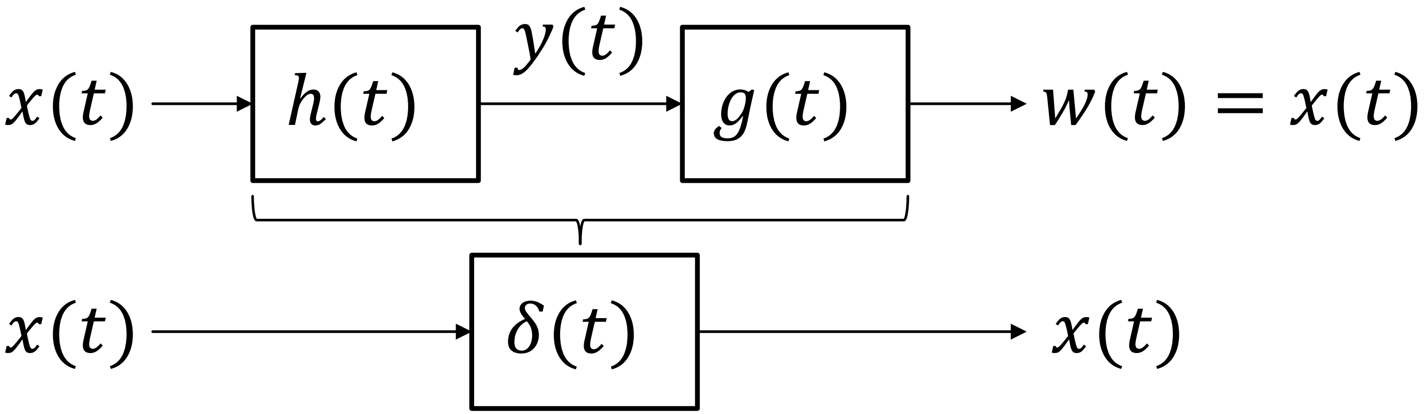
MEMORYLESS

- A system is memoryless if the output at any time depends only on input at the same time
- An LTI system is memoryless iff
  - $h(t) = a\delta(t) \quad h[n] = a\delta[n]$

- Half proof:
- If $h(t) = a\delta(t)$
- Then
  - $y(t) = \int_{-\infty}^{\infty} x(\tau) a\delta(t-\tau) d\tau = a \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau = ax(t)$

INVERTIBILITY

- The inverse of an LTI system must also be LTI
- An LTI system is invertible iff
  - There exists $g(t)$ such that $h(t) * g(t) = \delta(t)$



$$\left.\begin{aligned} y(t) &= x(t) * h(t) \\ w(t) &= y(t) * g(t) \end{aligned}\right\} \quad w(t) = x(t) * \underbrace{h(t) * g(t)}_{\delta(t)} = x(t)$$

CAUSALITY

- A LTI system is causal iff
  - $h(t) = 0 \quad t < 0 \qquad h[n] = 0 \quad n < 0$
- Half proof:
- Assume $h[k] = 0$ for $k < 0$
  - $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- Then
  - $y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$

STABILITY

- An LTI system is stable iff
  - $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$
  - Absolutely summable
- Half proof:
  - Given $|x(t)| < B \quad \forall t$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right|$$

triangle inequality

$$\leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|d\tau$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)| \underbrace{|x(t-\tau)|}_{\leq B} d\tau$$

$$\leq \underbrace{B \int_{-\infty}^{\infty} |h(\tau)|d\tau}_{\leq \infty}$$

STEP RESPONSE $s(t)$

- Discrete Time
- $s[n] = \sum_{k=-\infty}^{n} h[k]$
- $\delta[n] = u[n] - u[n-1]$
- $h[n] = s[n] - s[n-1]$
  - First difference

- Continuous Time
  - $s(t) = \int_{-\infty}^{t} h(\tau)d\tau$
- $h(t) = \frac{ds(t)}{dt}$
  - Derivative
- CT derivative property
  - Given $x(t) \rightarrow y(t)$
  - $\frac{dx(t)}{dt} \rightarrow \frac{dy(t)}{dt}$

EXAMPLE: CT DERIVATIVE PROPERTY

- LTI system output
  - $y(t) = \sin(\omega_0 t)$
- Input
  - $x(t) = e^{-5t}u(t)$
- Find impulse response

- Note
  - $\frac{dx(t)}{dt} = \frac{d}{dt}e^{-5t}u(t)$ product rule
  - $\frac{dx(t)}{dt} = e^{-5t}\delta(t) + (-5)e^{-5t}u(t)$
  - $\frac{dx(t)}{dt} = \delta(t) - 5x(t)$

- $\Rightarrow \delta(t) = \frac{dx(t)}{dt} + 5x(t)$

- Back to LTI system
- $\delta(t) \rightarrow h(t)$
- $\frac{dx(t)}{dt} + 5x(t) \rightarrow h(t)$
- $\frac{dx(t)}{dt} + 5x(t) \rightarrow \frac{dy(t)}{dt} + 5y(t)$
- $h(t) = \frac{dy(t)}{dt} + 5y(t)$
- $h(t) = \omega_0 \cos(\omega_0 t) + 5\sin(\omega_0 t)$

EXAMPLE: CT CONVOLUTION

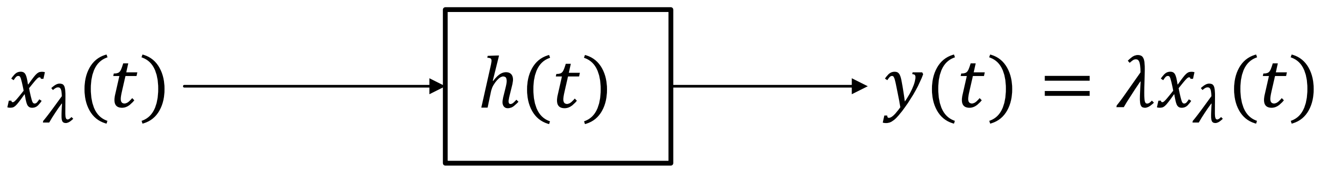
- $x(t) = h(t) = \begin{cases} 1 & -2 \leq t \leq 2 \\ 0 & else \end{cases}$
- Find output $y(t) = x(t) * h(t)$

CAUSAL LTI SYSTEMS DESCRIBED BY DIFFERENTIAL AND DIFFERENCE EQUATIONS

CHAPTER 2.4

EIGENFUNCTIONS OF LTI SYSTEMS

- Eigenfunction – a signal for which the LTI output is a constant times the input



  - λ is the eigenvalue (complex scalar)
- Turns out: (more in Chapter 3)
  - CT: $e^{st} \longrightarrow H(s)e^{st}$
  - DT: $z^n \longrightarrow H(z)z^n$

CAUSAL LTI DIFF EQ SYSTEMS

- It turns out that differential/difference equation relationships often occur in natural systems
- Need mathematical tools to study these systems effectively
  - This section will cover the typical approach from your previous math courses
    - Homogeneous + particular solutions
  - We will learn more effective Signals and Systems approach in the coming chapters

DIFFERENTIAL EQUATION LTI SYSTEMS

- $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$
  - N – highest derivative of $y(t)$
  - M – highest derivative of $x(t)$
  - a_k, b_k - constant coefficients
- Solution of the form:
  - $y(t) = y_p(t) + y_h(t)$
- Particular solution $y_p(t)$ satisfies diff equation above
- Homogeneous solution $y_h(t)$ satisfies
  - $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$
- Unique solution only when finding both $y_p(t)$ and $y_h(t)$ when using a set of auxiliary conditions (initial conditions)
  - $\frac{d^k y(t_0)}{dt^k}$ values for $k = 0, \ldots, N-1$
- Use exponentials to solve
  - $x(t) = e^{st} \rightarrow y(t) = H(s)e^{st} = \lambda x(t)$
  - $\frac{d}{dt} e^{st} = se^{st} = \lambda e^{st} = \lambda x(t)$

EXAMPLE: LTI DIFFERENTIAL SYSTEM

- Find solution to differential equation
- $\frac{dy(t)}{dt} + 2y(t) = x(t)$
- $x(t) = Ke^{3t}u(t)$

- Particular solution
  - Forced response – output is of the same form as input
  - $y_p(t) = Ax(t)$

- Homogeneous solution
  - Solution of the form

    $$y_h(t) = Be^{st}u(t)$$
  - s is an arbitrary unknown value that must be found

DEGENERATE DIFFERENTIAL EQ CASE

- For $N = 0$
  - $a_0 y(t) = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt} \Rightarrow y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt}$
- $y(t)$ is an explicit function of input $x(t)$
  - Given $x(t)$, can immediately get $y(t)$ by differentiation of $x(t)$

- Reminder for $N > 0$
  - Solve for $y(t) = y_p(t) + y_h(t)$
  - Given initial (rest) conditions: $y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1} y(t)}{dt^{N-1}} = 0$

DIFFERENCE EQUATION LTI SYSTEM

- $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$
- Same idea as CT case:
- Find $y[n] = y_p[n] + y_h[n]$
  - Choose form $y_h[n] = z^n$
  - Eigensignal for DT system

- Recursive difference eq form
- $y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right)$
  - Output at a time n can be computed from the current+past inputs and past output values
  - Need auxiliary eqs. To give past output initial conditions
    - E.g. values of $y[-1], y[-2], \ldots, y[-N]$

- Degenerate $N = 0$ case:
- $y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$
  - Non-recursive equation (no past output)
  - Only requires input signal
- This form matches convolutional form
  - $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
  - $h[n] = \begin{cases} b_n/a_0 & 0 \le n \le M \\ 0 & else \end{cases}$
- Known as a finite impulse response (FIR) system
  - Non-zero over a finite time interval

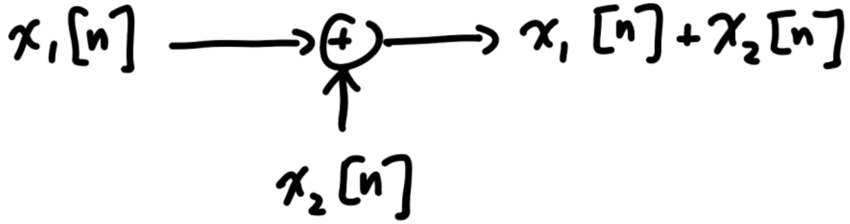
EXAMPLE: DT DIFF EQ SYSTEM

- Find output
- $y[n] - \frac{1}{2}y[n-1] = x[n]$
- Input impulse $x[n] = k\delta[n]$
- Condition of initial rest
  - Output does not change value until input changes
  - $y[n] = 0$ for $n < 0$

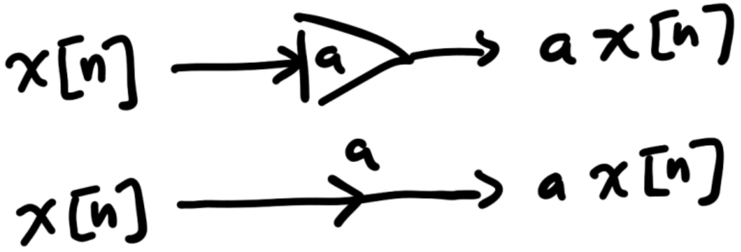
- Use recursive difference equation form to solve
- $y[n] = \frac{1}{2}y[n-1] + x[n]$
- Requires $y[n-1]$ to compute recursively

BLOCK DIAGRAMS FOR SYSTEMS

- Addition

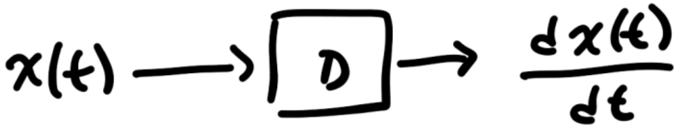


- Scaling

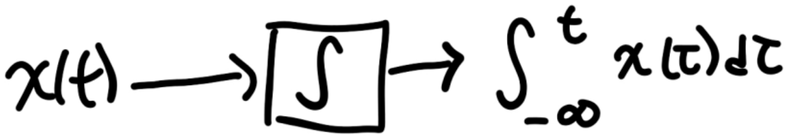


- Delay

- Differentiator

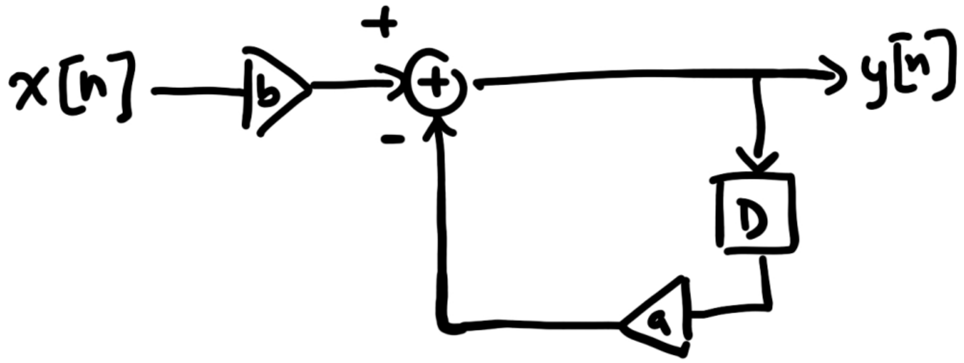


- Integrator

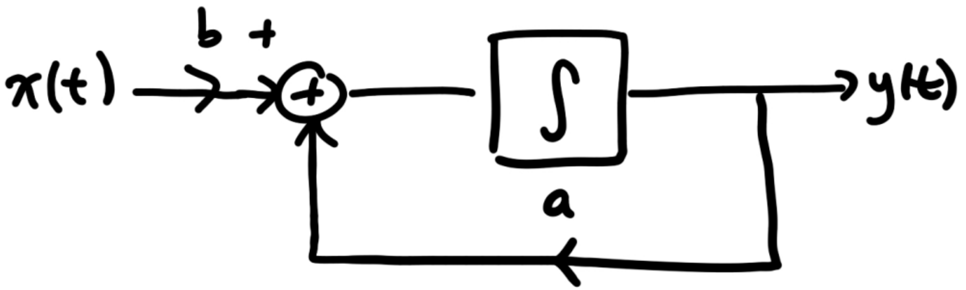


EXAMPLES: SIMPLE BLOCK DIAGRAMS

- $y[n] = -ay[n-1] + bx[n]$



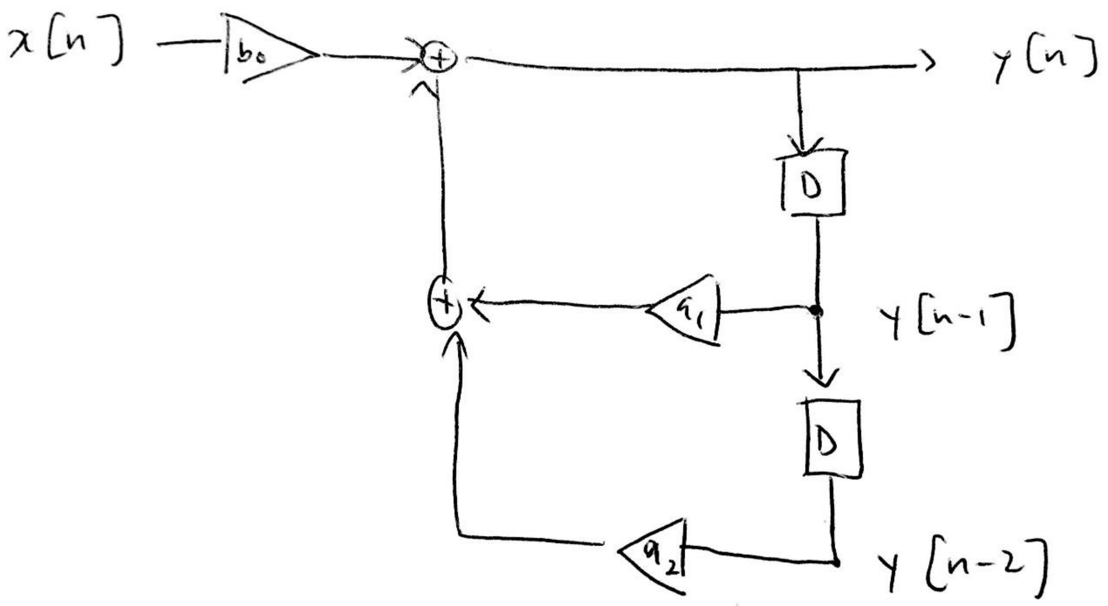
- $\frac{dy(t)}{dt} + ay(t) = bx(t)$
  - $y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$
- Preferred with integral
  - $\frac{dy(t)}{dt} = -ay(t) + bx(t)$
  - $y(t) = \int_{-\infty}^{t} \frac{dy(t)}{dt} dt$
  - $\quad = \int_{-\infty}^{t} [bx(t) - ay(t)] dt$



EXAMPLE: ANOTHER DT DIAGRAM

- $y[n] - a_1 y[n-1] - a_2 y[n-2] = b_0 x[n]$

- Rearrange
- $y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n]$



- Requires:
  - 3 multiplications
  - 2 additions
  - 2 delays (memory storage)

GENERAL DIFFERENCE EQUATION

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$ General diff eq.

$$a_0 y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- For simplicity, assume normalized coefficients $a_0 = -1$

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$ Recursive diff eq.

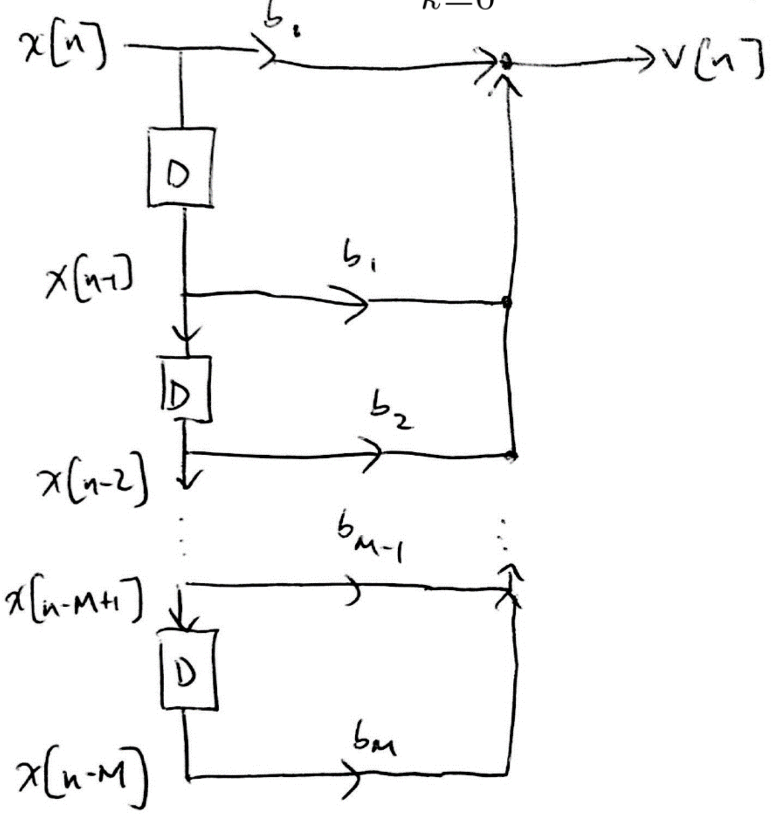
Note: a_k coefficients have opposite sign between recursive and general diff forms

DIRECT FORM I (DFI)

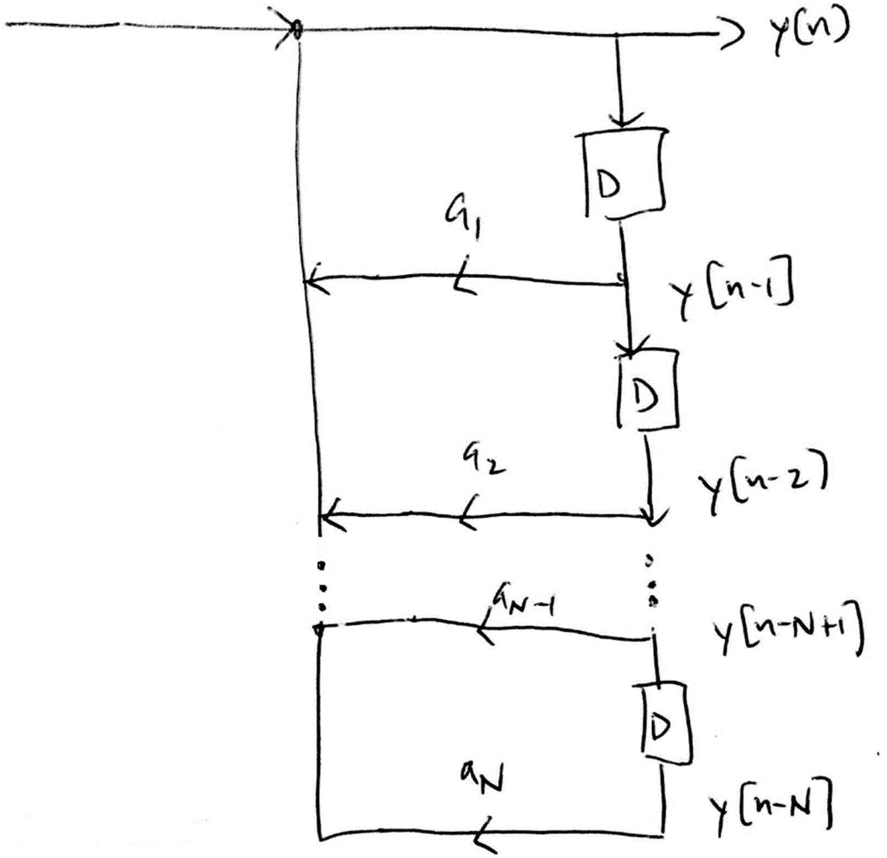
$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

Define $v[n] = \sum_{k=0}^{M} b_k x[n-k]$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + v[n]$$



Stack of M delays on input

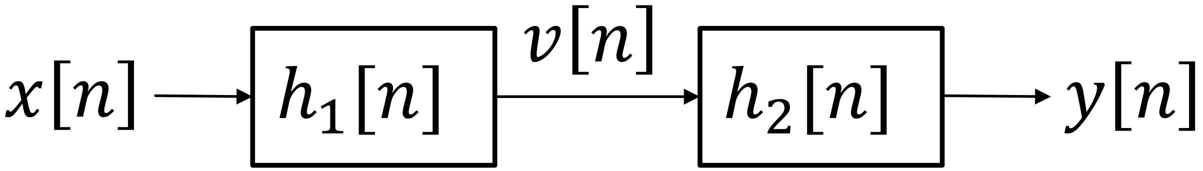


Stack of N delays on output

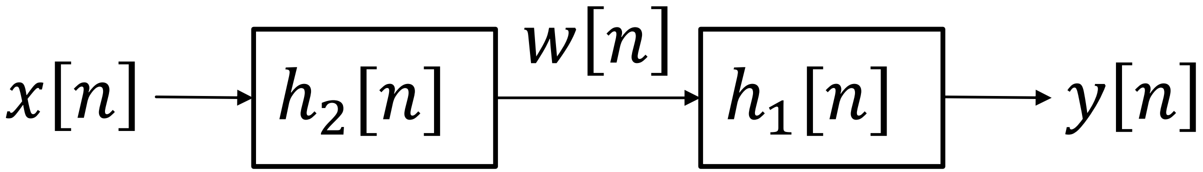
DF SUBSYSTEM CASCADE

- Notice DFI has two subsystems

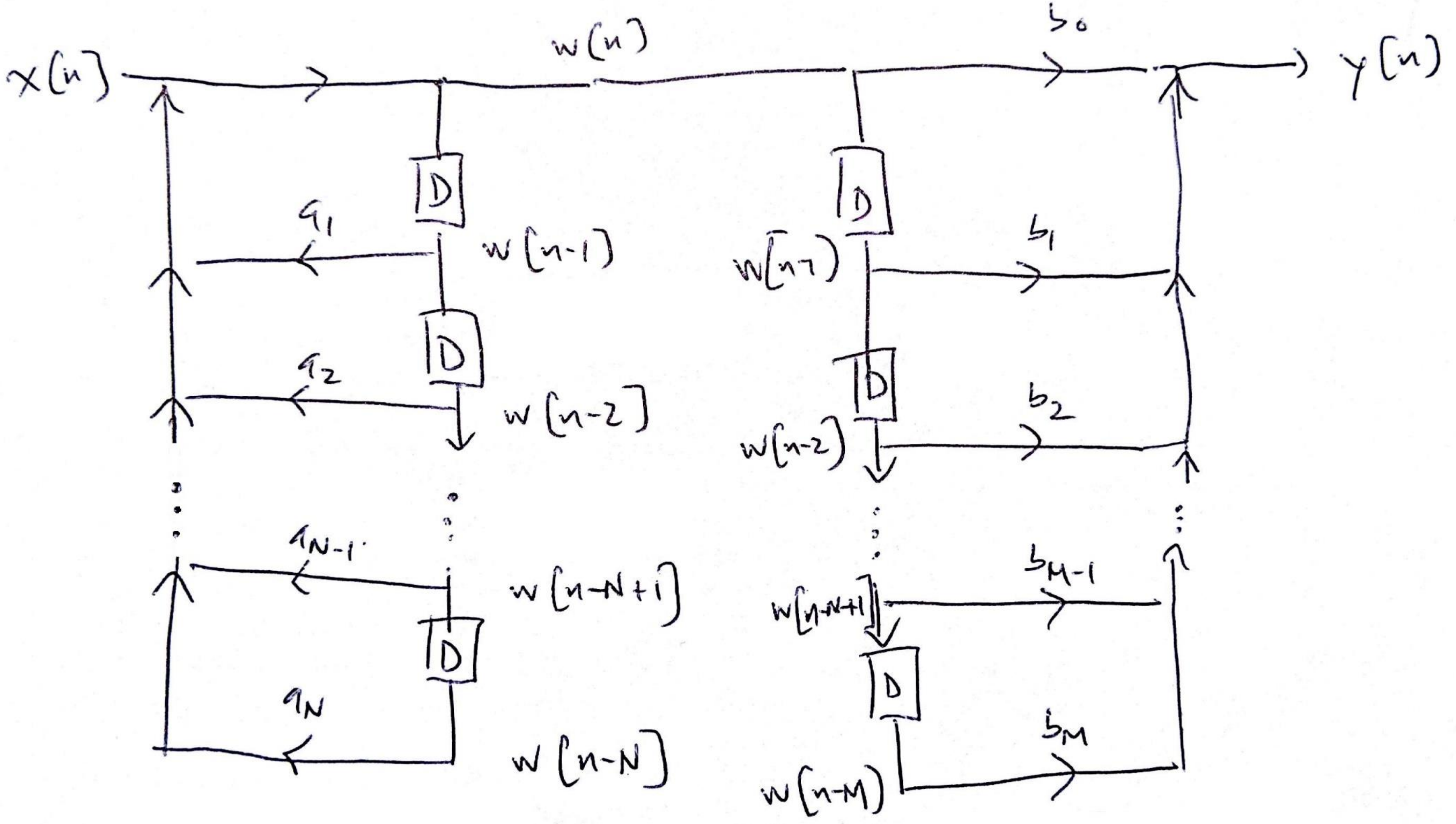
- Therefore



- Due to LTI system properties, can switch subsystems



DIRECT FORM – SWAP STACKS

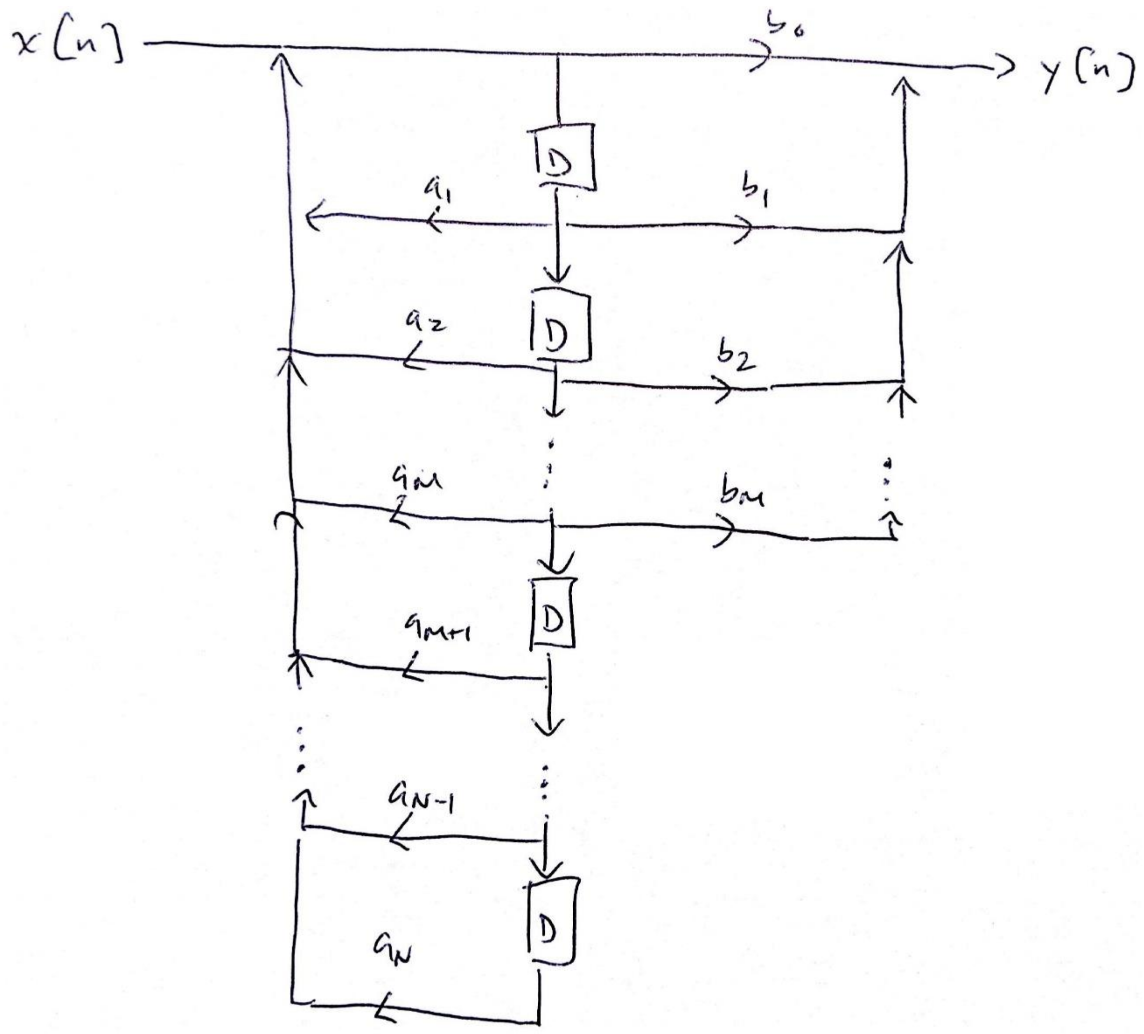


Stack of N delays on input

Stack of M delays on output

DIRECT FORM II (DFII) – DELAY SQUEEZE

- Notice: the delayed signal $w[n]$ is stored twice
- The diagram can be simplified
  - Assume $N > M$
- Canonical form
  - Minimize number of delays to max(N,M)
  - Min # multi – M+N+1
  - Min # adds (2 input) – M+ N

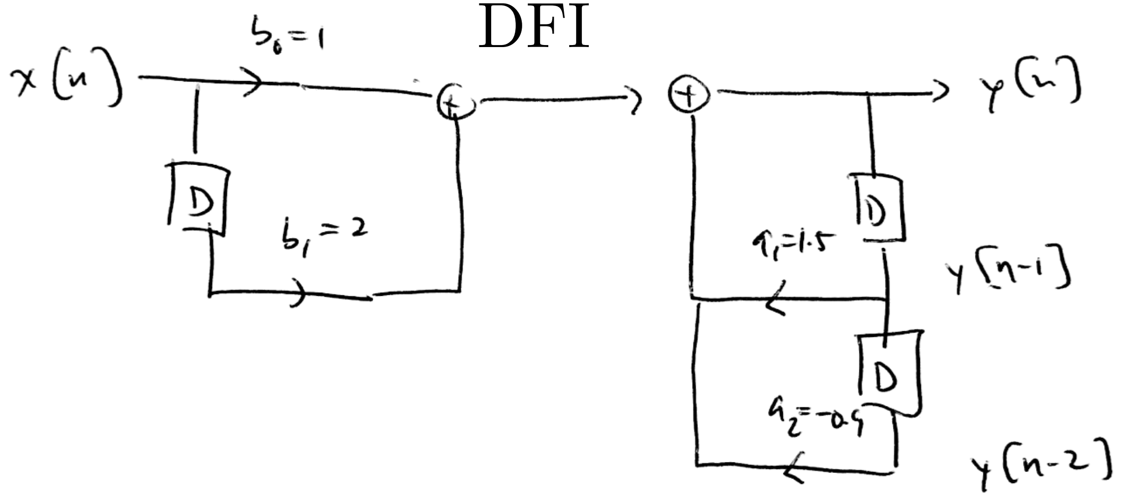


EXAMPLE: DFI, DFII

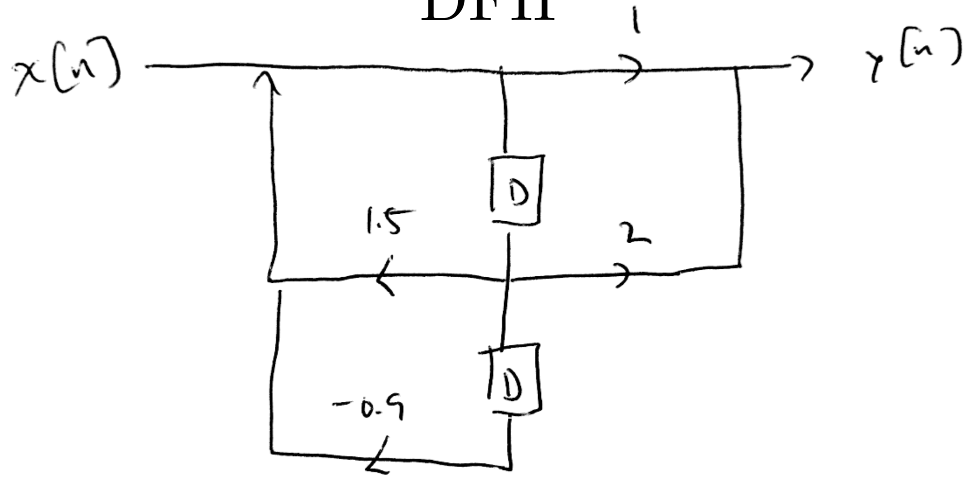
- Find DFI/DFII of following
- $y[n] - 1.5y[n-1] + 0.9y[n-2] = x[n] + 2x[n-1]$

$$y[n] = \underbrace{}_{b_0} x[n] + \underbrace{2}_{b_1} x[n-1] + \underbrace{1.5}_{a_1} y[n-1] \underbrace{-0.9}_{a_2} y[n-2]$$

DFI



DFII



Notice the feedback branches have opposite sign than in the general diff eq