EE360: SIGNALS AND SYSTEMS I

CH1: SIGNALS AND SYSTEMS

CONTINUOUS-TIME AND DISCRETE-TIME SIGNALS

CHAPTER 1.0-1.1

INTRODUCTION

- Signals are **quantitative** descriptions of physical phenomena
  - Represent a pattern of variation

EXAMPLE SIGNALS I

- Circuit

- v_s - voltage signal
- v_c - voltage signal
- i – current signal

- These are continuous-time signals

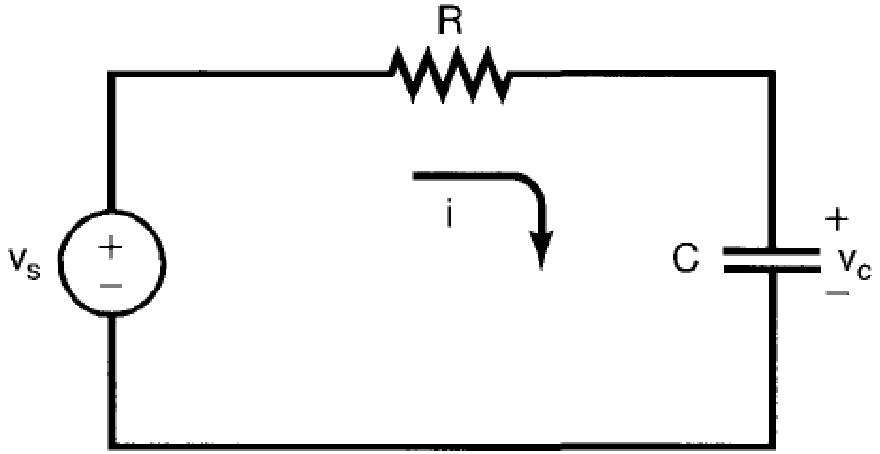


Figure 1.1 A simple *RC* circuit with source voltage v_s and capacitor voltage v_c.

EXAMPLE SIGNALS II

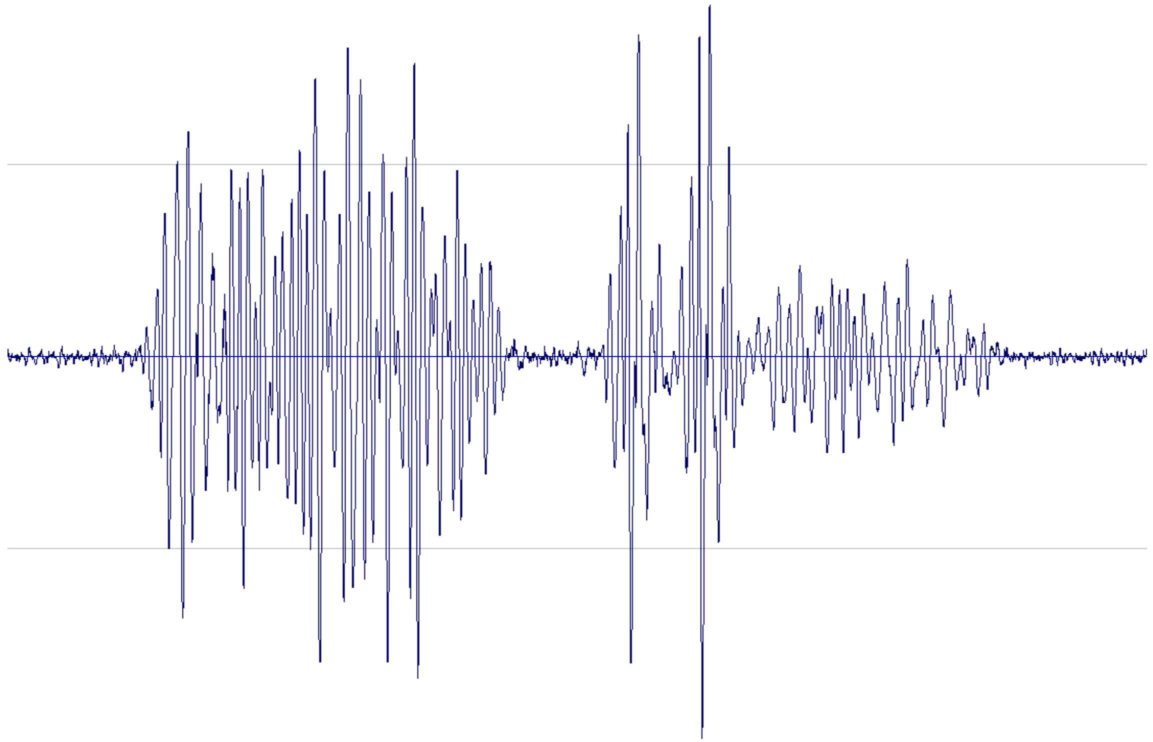
- Stock market price
- p – closing price signal
- Discrete time signal

EXAMPLE SIGNALS II

- Stock market price
- p – closing price signal
- Discrete time signal
- Tesla stock for fun
  - Last 3 months
  - Last 5 years

EXAMPLE SIGNALS III

- Audio signal
- Continuous signal in “raw” form
- Discrete signal when store on a CD/computer



MATHEMATICAL FORMULATION

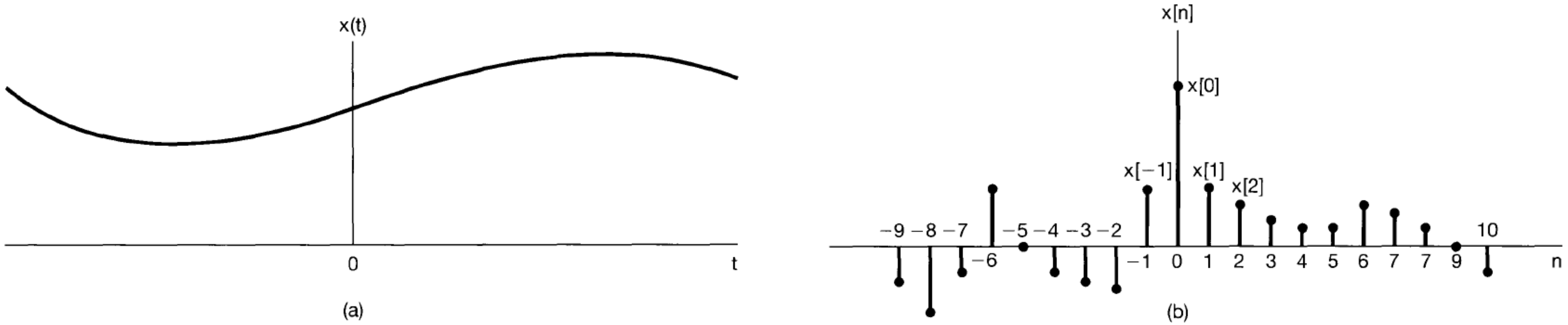
- In these examples, the signal is a function of one variable, **time**
  - $f(t) \longleftarrow$ focus of the book

- More generally, a signal can be a function of multiple variables and not just time
  - E.g. an image $I(x, y)$

SIGNAL TYPES

- This course deals with two types of signals
- Continuous-time (CT) signals
  - $x(t)$ with $t \in \mathbb{R}$ a real-values variable, denoting continuous time
    - Notice the parenthesis is used to denote a CT signal
- Discrete-time (DT) signals
  - $x[n]$ with $n \in \mathbb{Z}$ an integer-valued variable, denoting discrete time
    - Notice the square brackets to denote a DT signal
    - $x[1]$ is defined but $x[1.5]$ is not defined

GRAPHICALLY



- Note: $x(t)$ could signify the full signal or a value of the signal at a specific time t
  - May see $x(t_0)$ for a specific value of signal $x(t)$ when $t = t_0$ for clarity

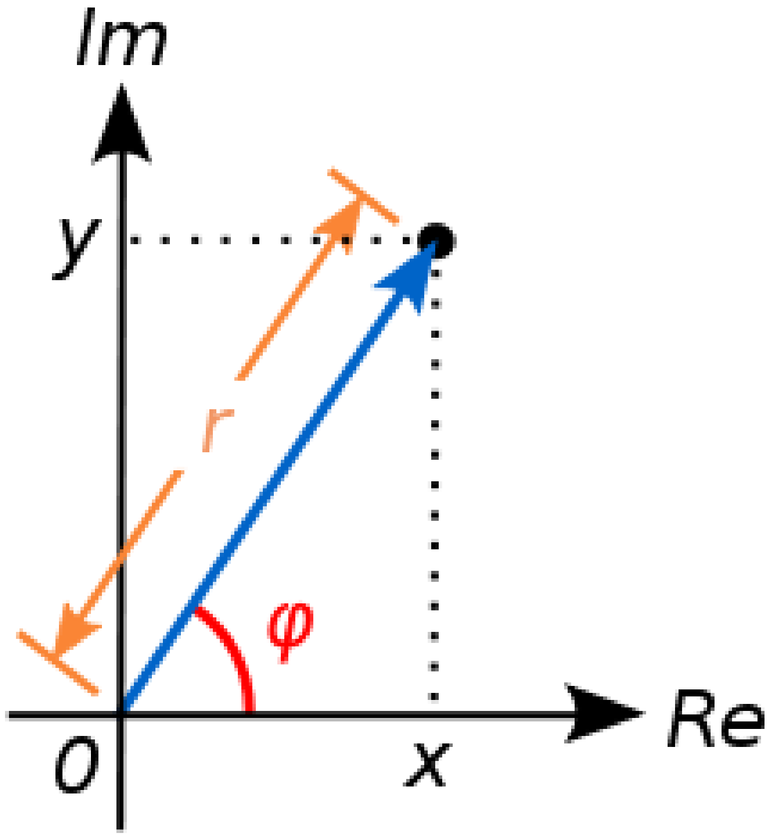
COMPLEX NUMBER REVIEW

- This course will often work with complex signals as they are mathematically convenient
  - $x(t) \in \mathbb{C}, \quad x[n] \in \mathbb{C}$
  - $\mathbb{C} = \{z | z = x + jy,\ x, y \in \mathbb{R}, j = \sqrt{-1}\}$

- Note the use of j for the imaginary number in electrical engineering rather than i

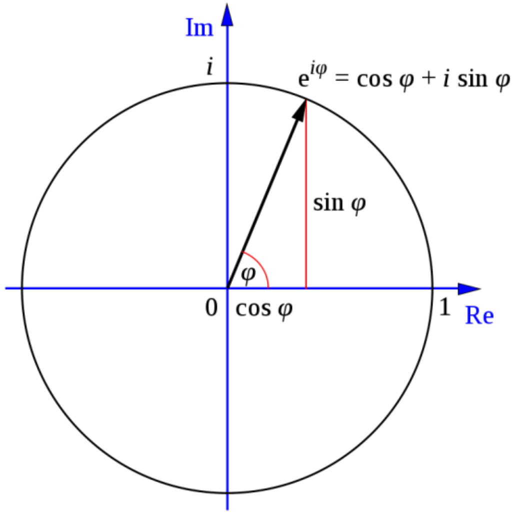
COMPLEX NUMBER REPRESENTATION

- Rectangular/Cartesian form
  - $z = x + jy$
  - $Re\{z\} = x$ real-part
  - $Im\{z\} = y$ imaginary-part
- Polar form
  - $z = re^{j\theta}$
  - $r^2 = x^2 + y^2$
  - $\theta = \arctan\left(\frac{y}{x}\right)$
  - $x = r\cos\theta$
  - $y = r\sin\theta$

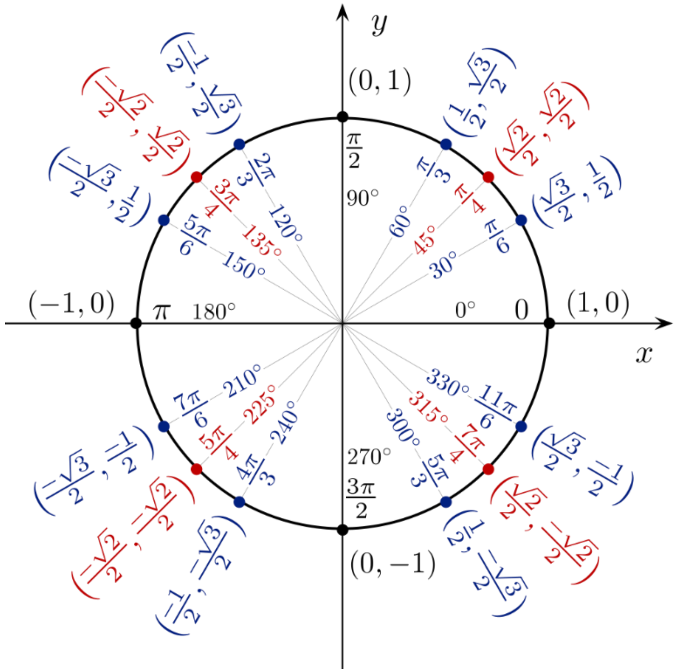


EULER'S FORMULA

- $e^{j\theta} = \cos\theta + j\sin\theta$
- Note:
  - $j = e^{j\pi/2}$ $\quad -1 = e^{j\pi}$
  - $-j = e^{j3\pi/2}$ $\quad 1 = e^{j2\pi k}$



- Know trig functions for common angles
  - For inverse trig function you must account for the quadrant



EXAMPLES: COMPLEX NUMBERS

- Express in polar form
  - $1 - j$
- Express in polar form
  - $(1 - j)^2$

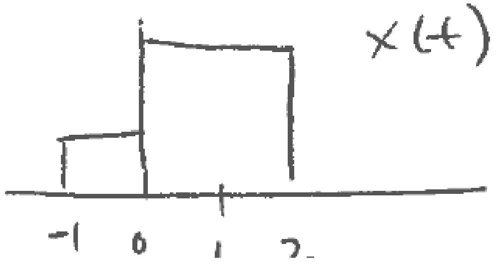
TRANSFORMATIONS OF THE INDEPENDENT VARIABLE

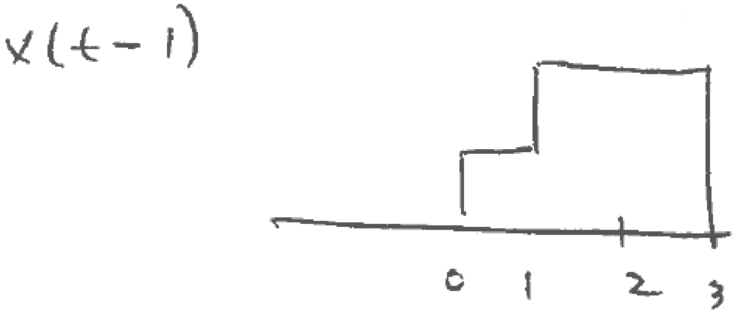
CHAPTER 1.2

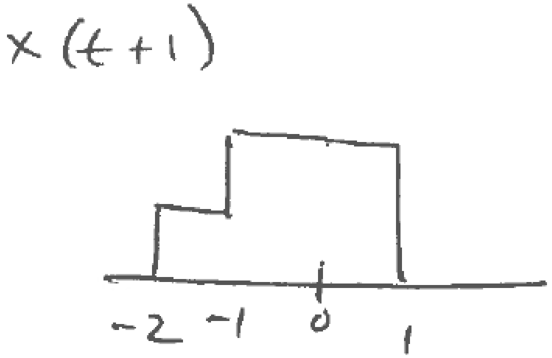
TIME SHIFT

- $x(t) \rightarrow x(t - t_0)$ $\quad\quad$ $x[n] \rightarrow x[n - n_0]$
  - $t_0 > 0 \Rightarrow$ delay $\quad\quad$ $t_0 < 0 \Rightarrow$ advance

example

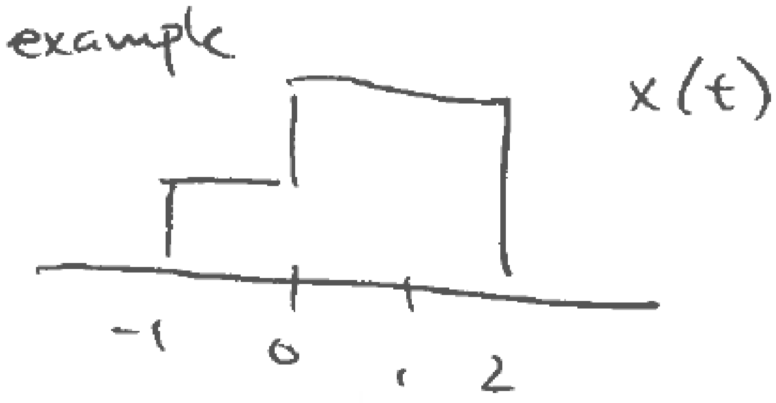


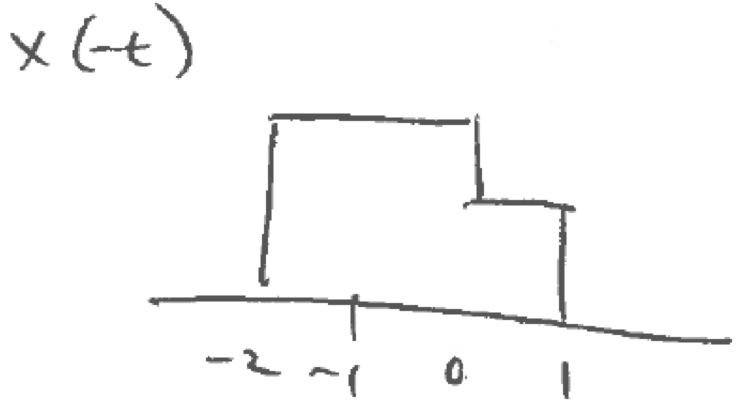




TIME REVERSAL

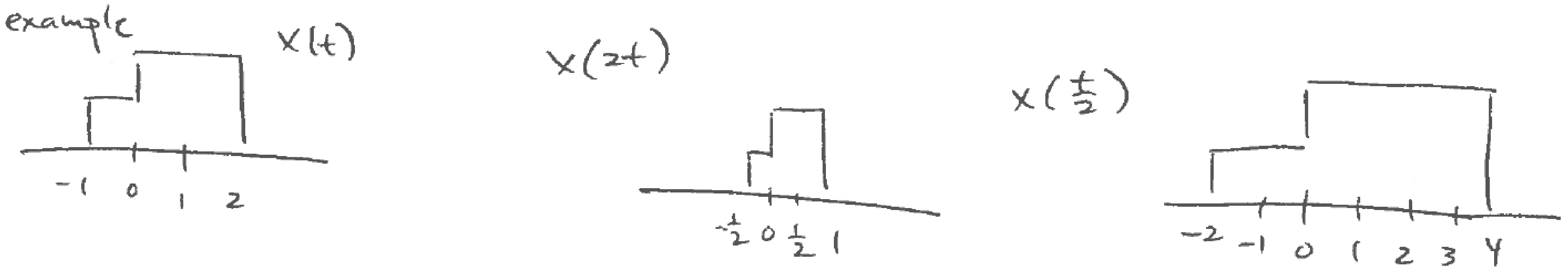
- $x(t) \rightarrow x(-t) \qquad x[n] \rightarrow x[-n]$
  - Flip signal across y-axis ($t = 0$ axis)





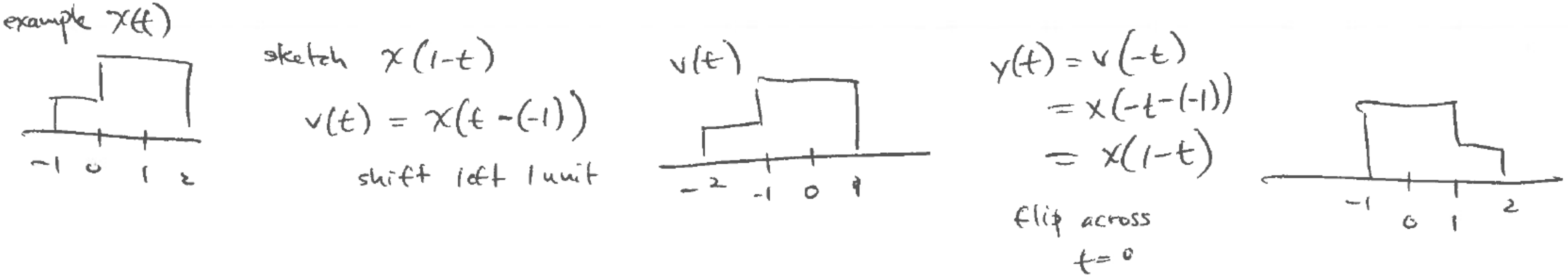
TIME SCALING

- $x(t) \rightarrow x(at) \quad a > 0$
  - $a > 1 \Rightarrow$ shrink time scale ("speed-up" or compress)
  - $0 < a < 1 \Rightarrow$ expand time scale ("slow-down" or stretch)
- $x[n] \rightarrow x[an] \quad a \in \mathbb{Z}^+$



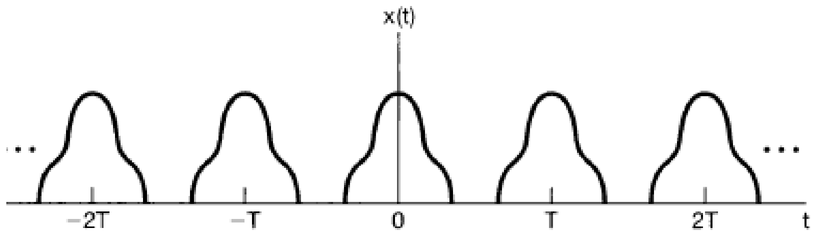
GENERAL TRANSFORMATION

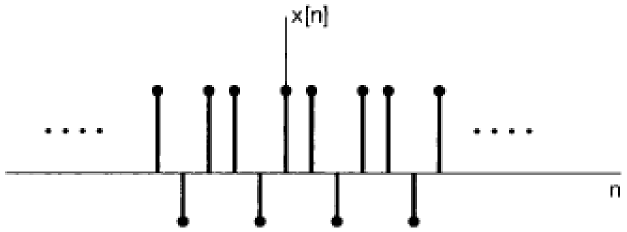
- $x(t) \rightarrow x(\alpha t - \beta)$ $\alpha < 0$ for time reversal
- General methodology – shift, then scale
  1. Shift: define $v(t) = x(t - \beta)$
  2. Scale: define $y(t) = v(\alpha t) = x(\alpha t - \beta)$
- Notice: scaling is only applied to time variable t



PERIODIC SIGNALS

- A signal is periodic if a shift of the signals leaves it unchanged
- Periodicity constraint
  - CT: there exists a $T > 0$ s.t.
    - $x(t) = x(t+T) \quad \forall t \in \mathbb{R}$
  - DT: there exists a $N > 0$ s.t.
    - $x[n] = x[n+N] \quad \forall N \in \mathbb{Z}$





FUNDAMENTAL PERIOD/FREQUENCY

- Note: $x(t) = x(t+T) = x(t+2T) = x(t+3T) = \cdots$
  - Periodic with period T or kT
- Fundamental period
  - T_0 is the fundamental period of $x(t)$ if it is the smallest value of $T > 0$ to satisfy the periodicity constraint (N_0 for DT)
- Fundamental frequency – inverse relationship to time
  - $\omega_0 = \frac{2\pi}{T_0}$ occasionally, $\Omega_0 = \frac{2\pi}{N_0}$
- Aperiodic signal – signal with no T, N satisfying periodicity constraint

EXAMPLES: FIND PERIOD

- $x(t) = e^{j\pi t/5}$
- $x[n] = e^{j\pi n/5}$

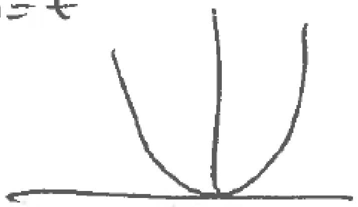
EVEN/ODD SIGNALS

- Even signal – same flipped across y-axis
  - $x[-n] = x[n]$
- Odd signal – upside-down when flipped
  - $x[-n] = -x[n]$
  - Note: must have $x[n] = 0$ at $n = 0$
- Decomposition theorem – any signal can be broken into sum of even and odd signals
  - $x(t) = y(t) + z(t), \quad y(t)$ even, $z(t)$ odd
- $y(t) = Ev\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$
- $z(t) = Odd\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$

examples

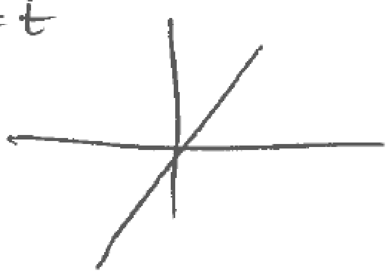
$x(t) = t^2$

even



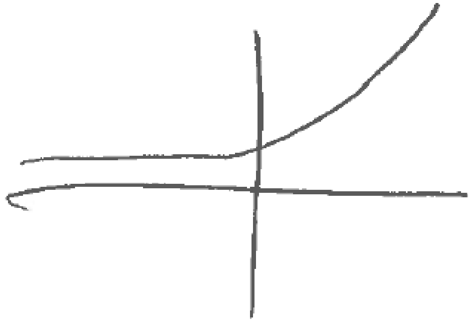
$x(t) = t$

odd



$x(t) = e^t$

neither



EXPONENTIAL AND SINUSOIDAL SIGNALS

CHAPTER 1.3

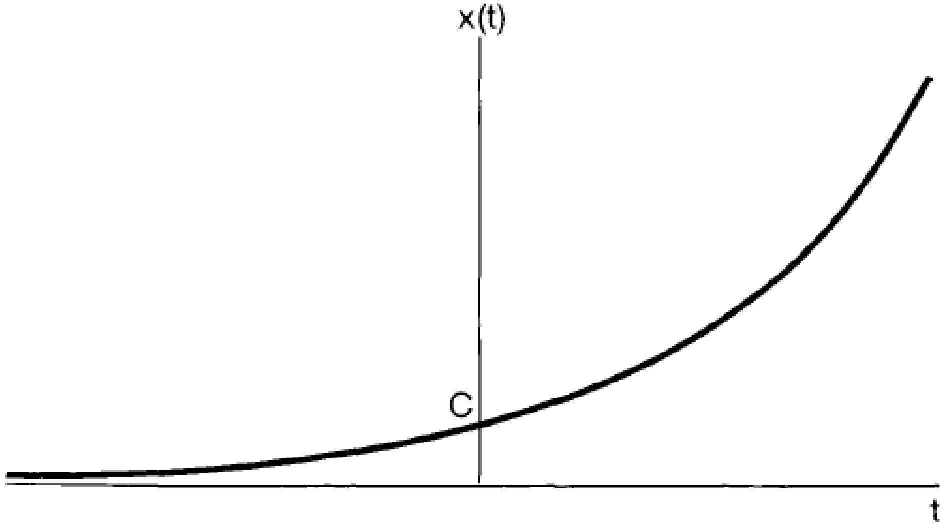
IMPORTANT CLASSES OF SIGNALS

1. Complex exponential - $Ce^{at}, Ce^{an} \quad C, a \in \mathbb{C}$
2. Impulse function - $\delta(t), \delta[n]$

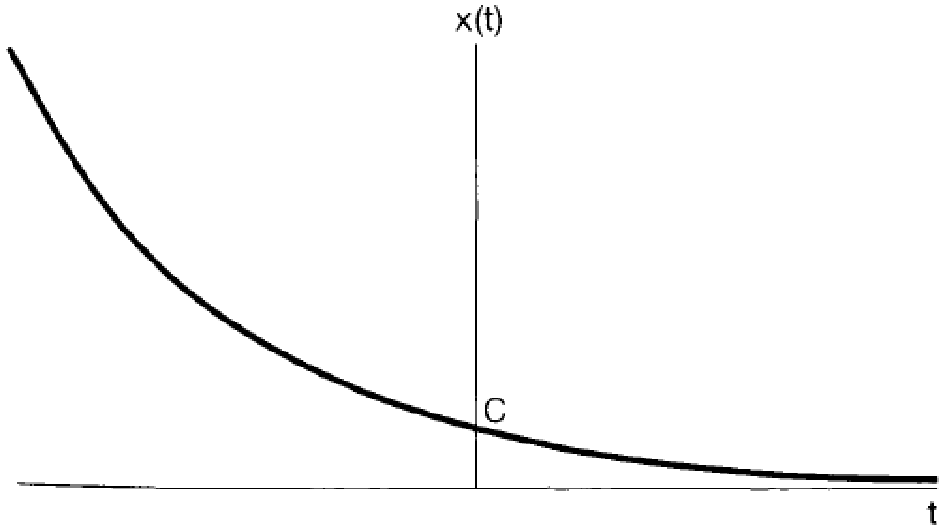
- Will want to represent general signals as linear combination of these special signals
  - The essence of linear system analysis
- Typically,
  - Impulse functions $\longrightarrow$ time-domain analysis
  - Complex exponentials $\longrightarrow$ frequency/transform domain analysis

REAL EXPONENTIAL SIGNALS

- $x(t) = Ce^{at} \quad C, a \in \mathbb{R}$



$a > 0$ exponential growth



$a < 0$ exponential decay

- $a = 0,\ x(t) = C$: constant function

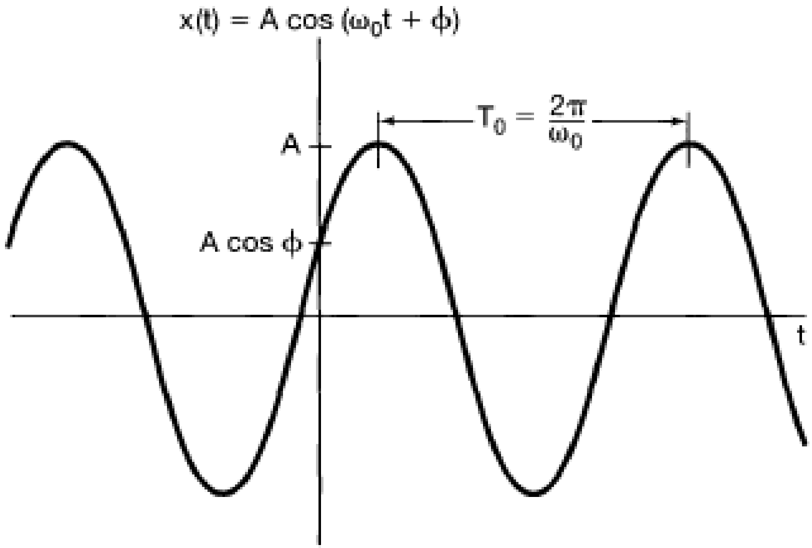
PERIODIC COMPLEX EXPONENTIAL

- $x(t) = Ce^{at}$
  - $a = j\omega_0, C = Ae^{j\theta}$
  - a is purely complex

$$x(t) = Ae^{j\theta}e^{j\omega_0 t} = Ae^{j(\omega_0 t+\theta)}$$
$$= \underbrace{A\cos(\omega_0 t + \theta)}_{\text{real}} + j\underbrace{A\sin(\omega_0 t + \theta)}_{\text{imaginary}}$$

- $x(t)$ is a pair of sinusoidal signals with the same amplitude A, frequency ω_0, and phase shift θ

- $Re\{x(t)\} = A\cos(\omega_0 t + \theta)$



- Proof

$$x(t) = x(t + T_0)$$
$$= Ce^{j\omega(t+2\pi/\omega_0)}$$
$$= Ce^{j\omega_0 t}\underbrace{e^{j2\pi}}_{=1} = Ce^{j\omega_0 t} = x(t)$$

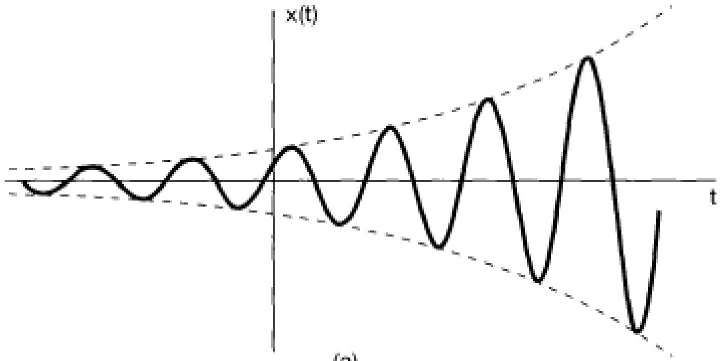
GENERAL COMPLEX EXPONENTIAL

- $x(t) = Ce^{at}$
  - $a = r + j\omega, C = Ae^{j\theta}$

$$x(t) = Ce^{at} = Ae^{j\theta}e^{(r+j\omega_0)t} = Ae^{rt}e^{j(\omega_0 t+\theta)}$$
$$= \underbrace{Ae^{rt}\cos(\omega_0 t+\theta)}_{\text{real}} + j\underbrace{Ae^{rt}\sin(\omega_0 t+\theta)}_{\text{imaginary}}$$

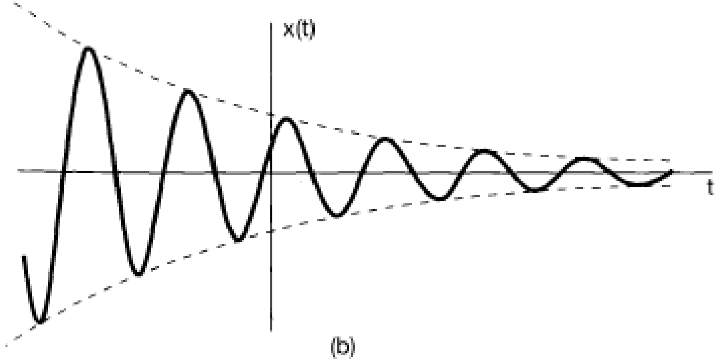
- Amplitude controlled sinusoid
  - Ae^{rt} defines envelope

- $r > 0$



(a)

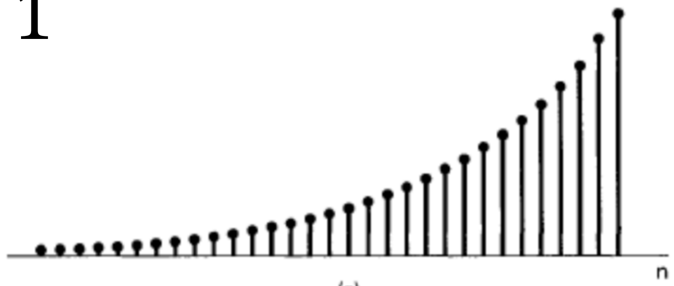
- $r < 0$



(b)

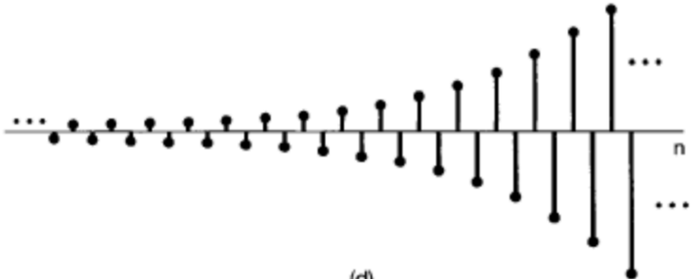
DT COMPLEX EXPONENTIAL - REAL

- $x[n] = Ce^{\beta n}$ or $x[n] = C\alpha^n$
  - $\alpha = e^{\beta}$, $C, \beta \in \mathbb{C}$
- $\alpha > 1$



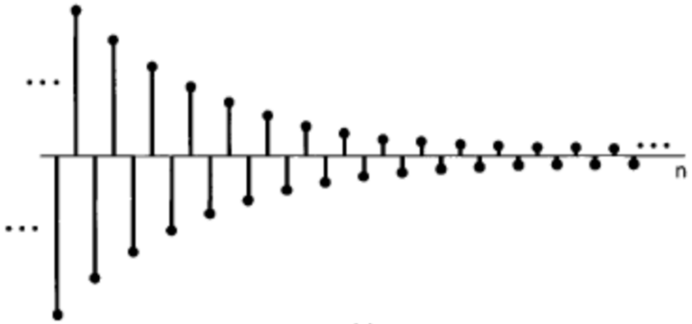
(a)

- $\alpha < -1$



(d)

- Real exponential
  - $C, \alpha \in \mathbb{R}$
- $-1 < \alpha < 0$



(c)

- $0 < \alpha < 1$

(b)

GENERAL DT COMPLEX EXPONENTIAL

- $x[n] = C\alpha^n, \quad C, \alpha \in \mathbb{C}$
  - $C = |C|e^{j\theta}, \ \alpha = |\alpha|e^{j\omega_0}$

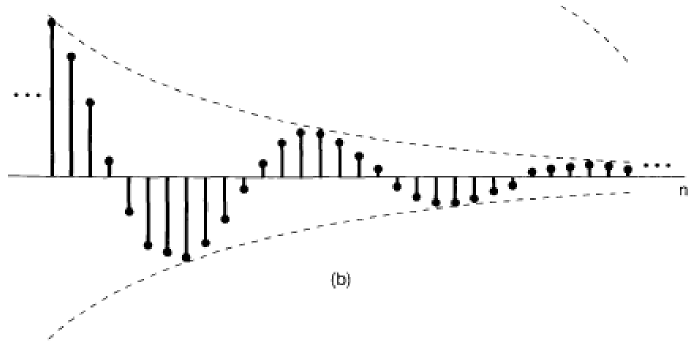
$$
\begin{aligned}
x[n] &= |C|e^{j\theta}\left(|\alpha|e^{j\omega_0}\right)^n \\
&= |C||\alpha|^n e^{j(\omega_0 n+\theta)} \\
&= |C||\alpha|^n \cos(\omega_0 n+\theta) + j|C||\alpha|^n \sin(\omega_0 n+\theta)
\end{aligned}
$$

- Three cases for $|\alpha|$
- $|a| = 1$

$$x[n] = |C|\cos(\omega_0 n+\theta) + j|C|\sin(\omega_0 n+\theta)$$

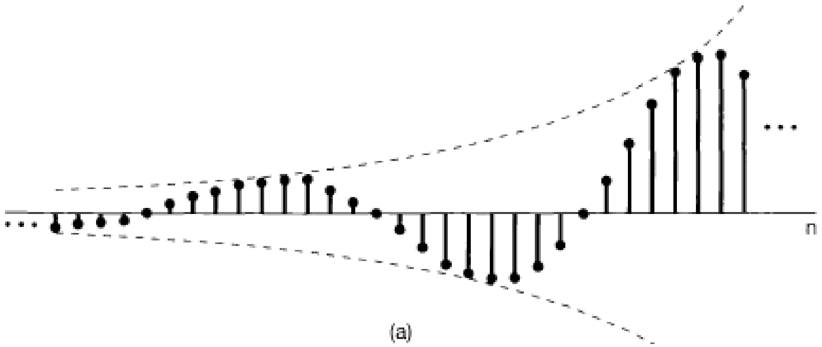
  - Not necessarily periodic

- $|\alpha| < 1$ - decaying exponential envelope



(b)

- $|\alpha| > 1$ - Growing exponential envelope



(a)

PERIODICITY OF DT COMPLEX EXPONENTIALS

- Unlike CT, there are conditions for periodicity
- Consider frequency $\omega_0 + 2\pi$
- $e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n}e^{j2\pi n} = e^{j\omega_0 n}$
  - Exponential with freq $\omega_0 + 2\pi$ is the same as exp. with freq ω_0
- → Only need to consider a 2π interval for ω_0
  - $0 \le \omega_0 \le 2\pi$ or $-\pi \le \omega_0 \le \pi$
  - See Fig. 1.27 of book

DT FREQUENCY RANGE

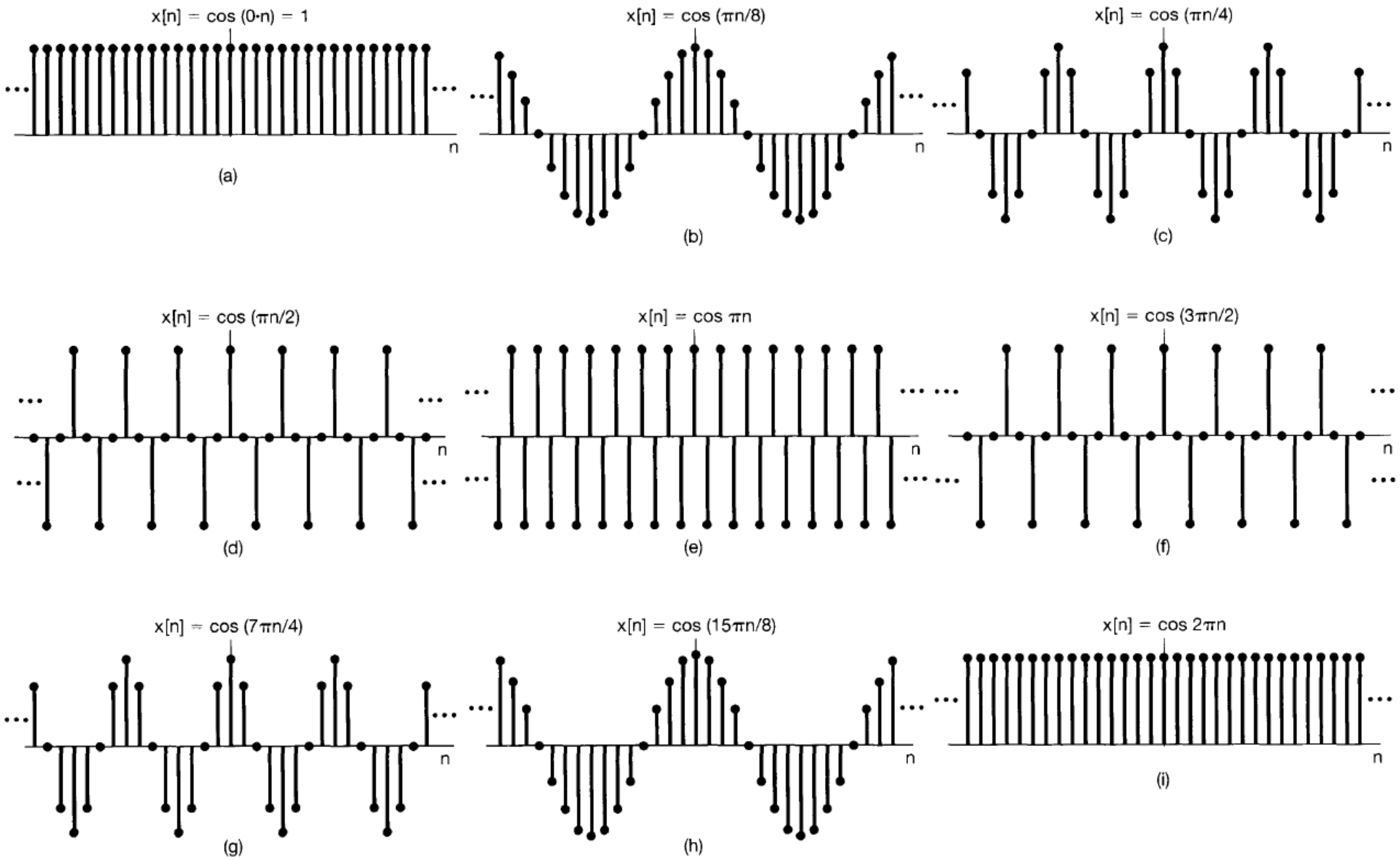


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

DT PERIODICITY CONSTRAINT

$$x[n] = x[n+N] \qquad \forall n \in \mathbb{Z}$$

$$e^{j\Omega_0 n} = e^{j\Omega(n+N)} = e^{j\Omega_0 n} e^{j\Omega_0 N}$$

$$\Rightarrow e^{j\Omega_0 N} = 1 = e^{j2\pi m} \qquad m \in \mathbb{Z}$$

$$\Rightarrow \Omega_0 N = 2\pi m$$

$$\Rightarrow \Omega_0 = \frac{2\pi m}{N}$$

- $e^{j\Omega_0 n}$ is periodic iff Ω_0 is a rational multiple of 2π
  - Fundamental period: $N = \dfrac{2\pi m}{\Omega_0}$
  - $\dfrac{m}{N}$ is in reduced form
    - $\gcd(m, N) = 1$ ← greatest common denominator

- Table 1.1 is good for highlighting the differences between DT and CT

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

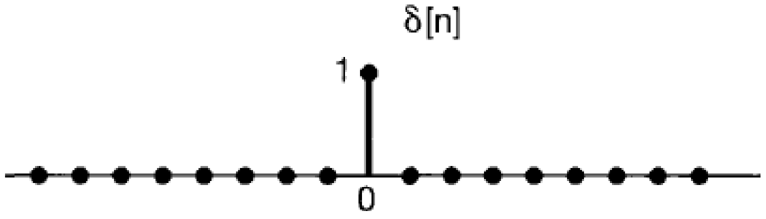
*Assumes that m and N do not have any factors in common.

THE UNIT IMPULSE AND UNIT STEP FUNCTIONS

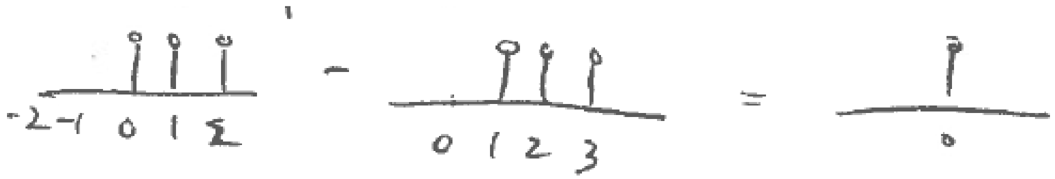
CHAPTER 1.4

DT IMPULSE AND UNIT STEP FUNCTIONS

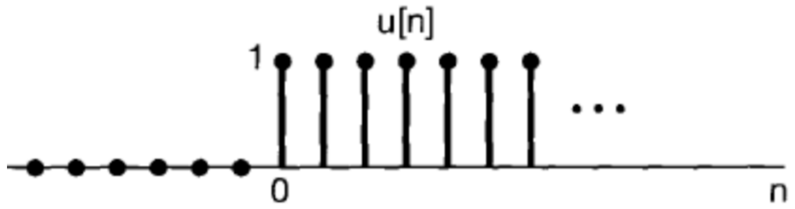
- Unit impulse (Kronecker delta)
  - $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$



- $\delta[n] = u[n] - u[n-1]$



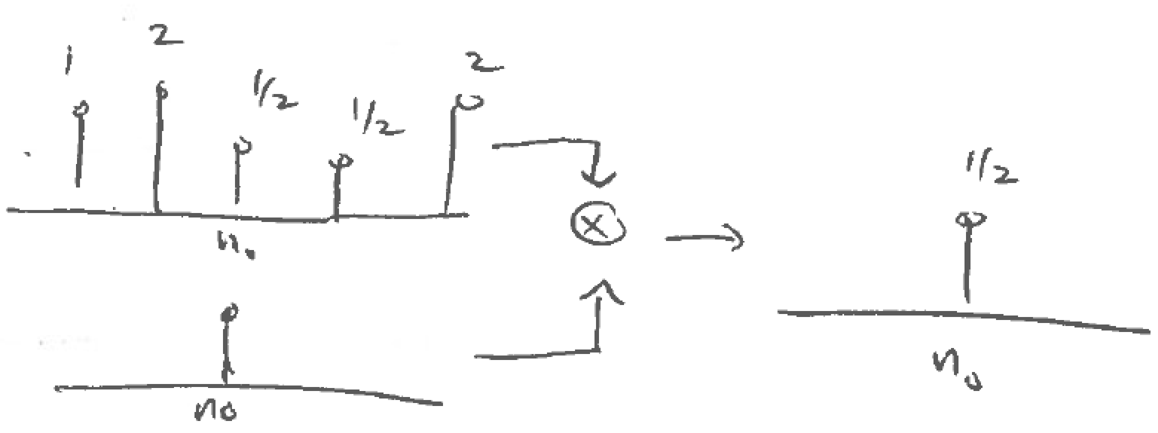
- Unit step
  - $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



- $u[n] = \sum_{m=-\infty}^{n} \delta[m]$
  - Running (cumulative) sum
- $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$
- $\quad = \sum_{k=-\infty}^{\infty} u[k]\delta[n-k]$
  - Sum of delayed impulses

SAMPLING/SIFTING PROPERTIES

- Sampling Property
  - $x[n]\delta[n] = x[0]\delta[n]$
  - $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$

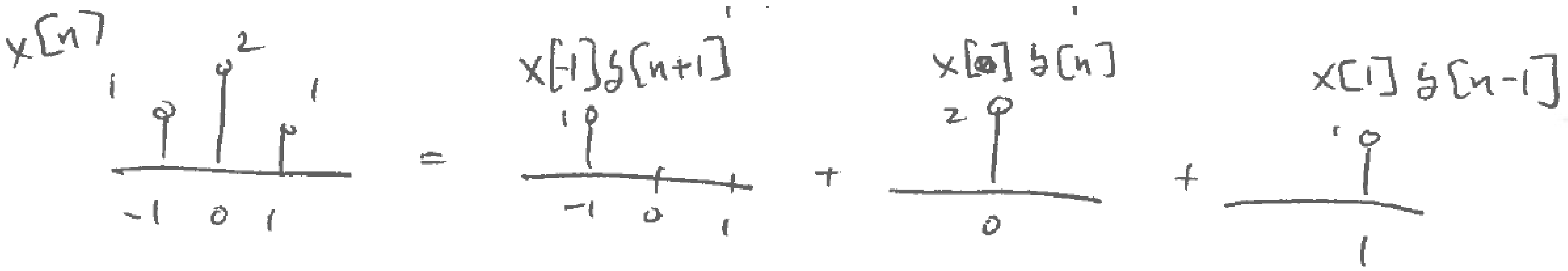


- Product of signals is a signal
  - Multiply values at corresponding time

- Sifting Property
  - $\sum_{m=-\infty}^{\infty} x[m]\delta[m] = x[0]$
  - $\sum_{m=-\infty}^{\infty} x[m]\delta[m-n_0] = x[n_0]$
- Notice above is summation of values in the sampled signal
- More generally, this summation holds for any limits that contain the impulse

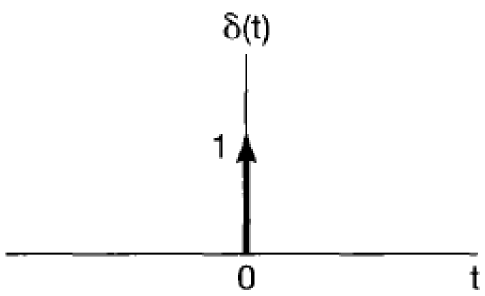
REPRESENTATION PROPERTY

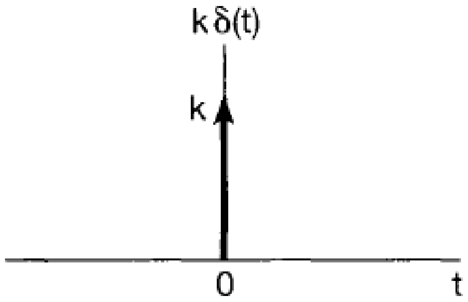
- Every DT signal can be represented as a linear combination of shifted impulses
- $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
  - $x[k]$ – value of signal at time k
  - A bit complicated but useful for study of LTI systems (Ch2)

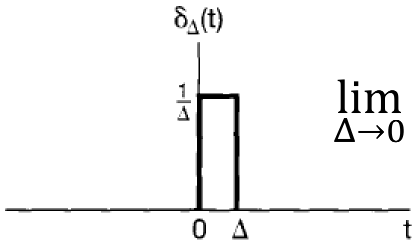


CT IMPULSE AND UNIT STEP FUNCTIONS

- Unit impulse (dirac delta)
  - $\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$
  - With $\int_{-\infty}^{\infty} \delta(t)dt = 1$

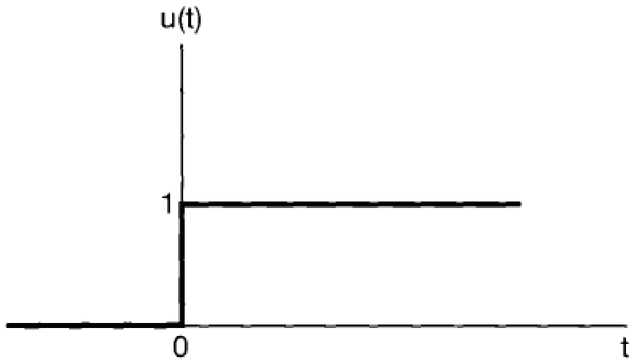






$\lim_{\Delta \to 0}$

- Unit step
  - $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$



- Relationship
  - $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$
  - $\delta(t) = \frac{du(t)}{dt}$

PROPERTIES

- Sampling
  - $x(t)\delta(t) = x(0)\delta(t)$
  - $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
  - Product of two signals is a signal
- Representation property
  - $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$
- Example
  - $u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t - \tau)d\tau$
- Sifting

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = \int_{-\infty}^{\infty} x(0)\delta(t)dt$$
$$= x(0)\underbrace{\int_{-\infty}^{\infty} \delta(t)dt}_{=1}$$
$$= x(0)$$

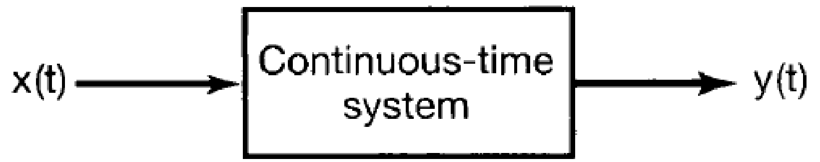
$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$$

CONTINUOUS-TIME AND DISCRETE-TIME SYSTEMS

CHAPTER 1.5

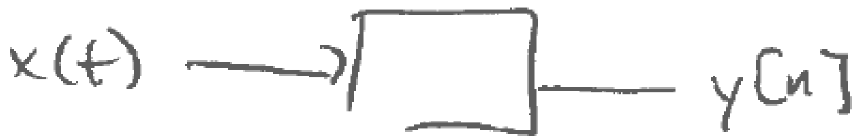
SYSTEMS

- A system is a quantitative description of a physical process to transform an input signal into an output signal
  - Systems are a black box – a mathematical abstraction

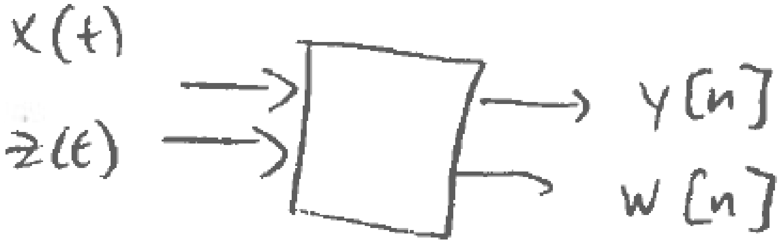


(a)

- Shorthand notation
  - $x(t) \longrightarrow y(t)$
- More complex systems
  - Sampling

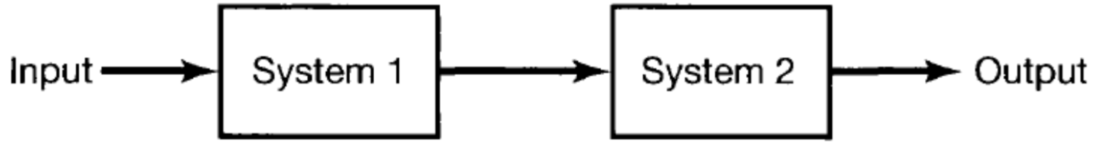


  - MIMO (multi input/multi output

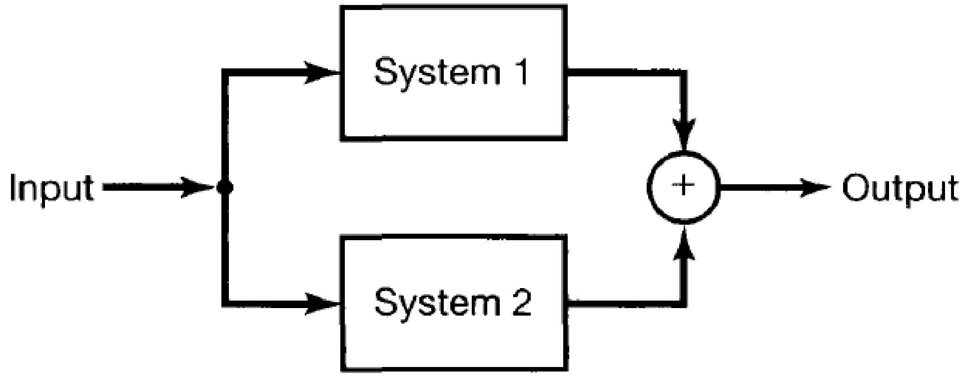


SYSTEM INTERCONNECTIONS

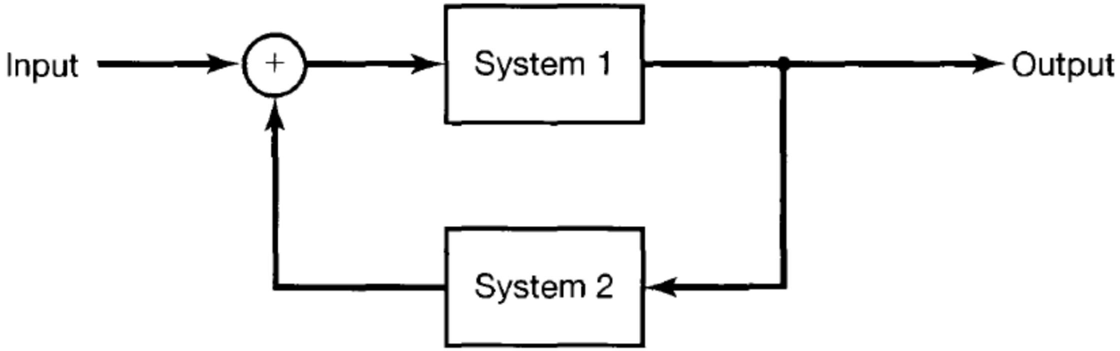
- Series/cascade connection



- Parallel interconnection



- Feedback connection
  - Very important in controls



- More complex systems can be composed by various series/parallel interconnections

BASIC SYSTEM PROPERTIES

CHAPTER 1.6

BASIC SYSTEM PROPERTIES

- Memoryless
- Invertibility
- Causality
- Stability
- Linearity
- Time-invariance

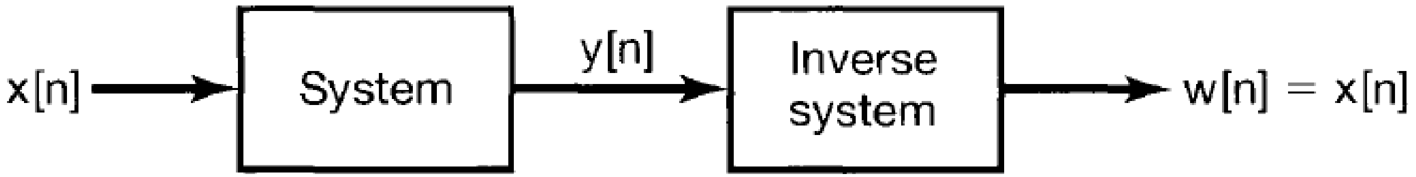
Define an important class of systems called LTI

MEMORYLESS SYSTEMS

- A system is memoryless if the output at a time t depends only on input at the same time t
- Examples
  - $y(t) = \left(2x(t) - x^2(t)\right)^2$
  - $y[n] = x[n]$
  - $y[n] = x[n-1]$
  - $y[n] = x[n] + y[n-1]$

INVERTIBILITY

- A system is invertible if distinct inputs lead to distinct outputs



- Rules for proving invertible systems
  - Show invertible by given the inverse system expression/formula
  - Show non-invertible by any counter example

EXAMPLES: INVERSE SYSTEMS

- $y(t) = (\cos t + 2)x(t)$
  - $x(t) = \frac{y(t)}{\cos t + 2}$
  - Invertible (no divide by zero!)
- $y[n] = \sum_{k=-\infty}^{n} x[k]$
  - $\Rightarrow y[n] = x[n] + y[n-1]$
  - $\Rightarrow x[n] = y[n] - y[n-1]$
  - Invertible
- $y(t) = x^2(t)$
  - $x_1(t) = 1 \Rightarrow y_1(t) = 1$
  - $x_2(t) = -1 \Rightarrow y_2(t) = 1$
  - Need unique input → distinct output
  - Not invertible

CAUSALITY

- A system is causal if the output at any time t depends only on the input at same time t or past times $\tau < t$
- Real systems must be causal because we cannot know future values
  - Buffering gives the appearance of non-causality
- Examples
  - $y[n] = x[n]$
  - $y[n] = x[n] + x[n+1]$
  - $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
  - $y[n] = x[-n]$
  - $y(t) = x(t)(\cos(t+2))$

STABILITY

- A system is stable if a bounded input results in a bounded output signal → BIBO stable
- A signal is bounded if there exists a constant B such that
  - $|x(t)| \leq B \quad \forall t$ and $B < \infty$
- BIBO condition
  - $|x(t)| \leq B \longrightarrow |y(t)| < \infty$

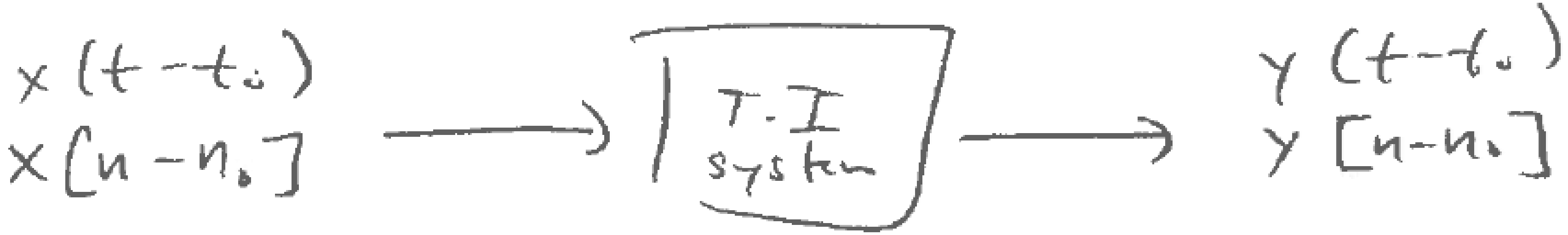
EXAMPLES: BIBO STABILITY

- $y(t) = 2x^2(t-1) + x(3t)$

$$y[n] = \begin{cases} 0 & n < 0 \\ 1.01y[n-1] + x[n] & n \geq 0 \end{cases}$$

TIME INVARIANCE

- A system is time-invariant if a time shift in input signal results in an identical time shift in the output signal



- Steps to check for TI
  - Assume $x(t) \longrightarrow y(t)$

1. Check $y_1(t) = y(t - t_0)$ time shift on output
2. Check $y_2(t) = f(x(t - t_0))$ operate on time shifted input
3. Verify $y_1(t) = y_2(t)$ for TI

EXAMPLES: TIME INVARIANT SYSTEMS

- $y(t) = \sin\big(x(t)\big)$
- $y[n] = nx[n]$

LINEARITY

- A system is linear if it is additive and scalable
- If $x_1(t) \longrightarrow y_1(t)$ and $x_2(t) \longrightarrow y_2(t)$
  - Additive
    - $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$
  - Scalable
    - $ax_1(t) \longrightarrow ay_1(t) \qquad a \in \mathbb{C}$
- Then, $ax_1(t) + bx_2(t) \longrightarrow ay_1(t) + by_2(t)$

EXAMPLES: LINEAR SYSTEMS

- $y(t) = tx(t)$
- $y[n] = 2x^2[n]$