EE360: SIGNALS AND SYSTEMS I

CH10: Z-TRANSFORM

INTRODUCTION

CHAPTER 10.0

INTRODUCTION

- Previously we saw the Laplace Transform
  - Extension of FS → FT → LT
  - Allowed us to study a wide class of signals/systems (unstable systems with ROC)
- The Z-Transform is the discrete version
  - While very similar, must recognize the specific differences

EIGENSIGNAL BACKGROUND

- Remember

$$x[n] = z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \underbrace{H(z)}_{\text{eigenvalue}} z^n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

THE Z-TRANSFORM

CHAPTER 10.1

Z-TRANSFORM DEFINITION

- The eigensignal result leads to the definition of the Z-Transform

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Shorthand notation

$$x[n] \overset{Z}{\longleftrightarrow} X(z)$$

FOURIER TRANSFORM CONNECTION

- Previously with Laplace, we saw the LT reduced to the FT along the $j\omega$-axis (stability constraint)
- For the Z-Transform, it reduces to the FT along the $e^{j\omega} = 1$ unit circle
- When $z = e^{j\omega}$
  - $X(z)|_{z=e^{j\omega}} = X(e^{j\omega}) = \Im\{x[n]\}$
  - $\Im\{.\}$ is the Fourier Transform

EXAMPLE 10.1

- Find the Z-transform of input $x[n] = a^n u[n]$

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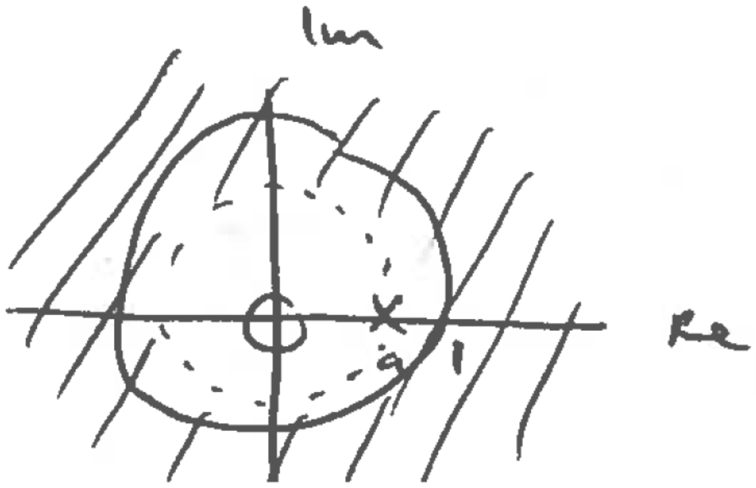
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$
$$= \sum_{n=0}^{\infty} \left(a z^{-1}\right)^n = \sum_{n=0}^{\infty} \alpha^n$$
$$= \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$



Note: for sum convergence

$$|\alpha| < 1 \Rightarrow |a z^{-1}| < 1$$

$$\text{ROC: } |z| > |a|$$

- Note: with z^{-1} we get a pole and a zero

EXAMPLE 10.2

- Find the Z-transform of input $x[n] = -a^n u[-n-1]$

EXAMPLE 10.3

- Find the Z-transform of input $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$

EXAMPLE 10.4

- Find the Z-transform of input $x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n]$

RATIONAL $X(z)$

- When $X(z)$ is a ratio of polynomials (from difference equation) there is:
  - Pole @ ∞ when the degree of the numerator exceeds the denominator
  - Zero @ ∞ when the numerator is of smaller degree than the denominator
- Must have balance (equal number of poles and zeros)

THE REGION OF CONVERGENCE FOR THE Z-TRANSFORM

CHAPTER 10.2

9 ROC PROPERTIES I

1. The ROC consists of rings in the z-plane centered about the origin
2. The ROC does not contain any poles
3. When $x[n]$ is finite duration, the ROC is the entire z-plane
   - Except possibly $z = 0$ and/or $z = \infty$ (poles @ zero and/or ∞)

9 ROC PROPERTIES II

4. When $x[n]$ is a right-sided sequence, if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC
5. When $x[n]$ is a left-sided sequence, if the circle $|z| = r_0$ is in the ROC, then $0 < |z| < r_0$ will also be in the ROC
6. When $x[n]$ is a two-sided sequence, if the circle $|z| = r_0$ is in the ROC, then the ROC will be a ring in the z-plane that includes $|z| = r_0$

9 ROC PROPERTIES III

7. If $X(z)$ is rational, then the ROC is bounded by poles or extends to infinity
8. If $X(z)$ is rational and right-sided, then the ROC is outside the outermost pole
   - If $x[n]$ is also causal, the ROC also includes $z = \infty$
9. If $X(z)$ is rational and left-sided, then the ROC is inside the innermost pole (not including poles @ z=0)
   - If $x[n]$ is also anticausal ($x[n] = 0 \ \forall n > 0$), the ROC also includes $z = 0$

EXAMPLE 10.8

- List all possible ROC for

$$X(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)(1 - 2z^{-1})}$$

INVERSE Z-TRANSFORM

CHAPTER 10.3

INVERSE Z-TRANSFORM

- Definition
  - $x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
- This is a contour integral within the ROC
- Like with LT, will avoid solving this directly and instead use
  - Inspection method (PFE + known pairs [Table 10.2 pg 776])
  - Power series expansion

EXAMPLE 10.10

- Find the inverse of

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

ROC: $\frac{1}{4} < |z| < \frac{1}{3}$

EXAMPLE 10.10

- Find the inverse of

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

ROC: $\frac{1}{4} < |z| < \frac{1}{3}$

- Do PFE and associate ROCs

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}} \longleftrightarrow x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

$|z| > \frac{1}{4}$ (right-sided) $\qquad |z| < \frac{1}{3}$ (left-sided)

POWER SERIES EXPANSION

- For finite sequences, can read of $x[n]$ directly by the z-power (useful for non-rational z-transform)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \ldots + x[-1]z^{1} + x[0] + x[1]z^{-1} + \ldots$$

EXAMPLE 10.12

- Find inverse of

$$X(z) = 4z^2 + 2 + 3z^{-1} \qquad\qquad 0 < |z| < \infty$$

EXAMPLE 10.12

- Find inverse of

$$X(z) = 4z^2 + 2 + 3z^{-1} \qquad 0 < |z| < \infty$$

$$= x[-2]z^2 + x[0]z^0 + x[1]z^{-1}$$

$$x[n] = \begin{cases} 4 & n = -2 \\ 2 & n = 0 \\ 3 & n = 1 \end{cases} = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

GEOMETRIC EVALUATION OF THE FT

CHAPTER 10.4

Z-TRANSFORM PROPERTIES AND PAIRS

CHAPTER 10.5-10.6

PROPERTIES OF Z-TRANSFORM

- Same idea as for LT
- Time Shifting
  - $x[n-n_0] \leftrightarrow z^{-n_0}X(z)$
  - ROC $= R$ (with potential addition or deletion of origin or infinity)
- Convolution
  - $x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z)$
  - ROC $\supset R_1 \cap R_2$
- Differentiation in z-Domain
  - $nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$
  - ROC $= R$
- Will rely heavily on time-shift (diff-eq) and convolution
- See Table 10.1 for more properties

COMMON Z-TRANSFORM PAIRS

- Will very rarely compute z-transform directly from summation definition

- Bookmark:
  - Table 10.1 Properties of Z-Transform [pg 775]
  - Table 10.2 Transform Pairs [pg 776]

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

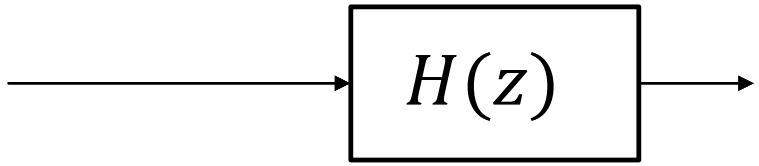
ANALYSIS AND CHARACTERIZATION OF LTI SYSTEMS USING Z-TRANSFORMS

CHAPTER 10.7

LTI SYSTEMS AND Z-TRANSFORMS

- By convolution property

$x[n]$



$y[n] = x[n] * h[n]$

$X(z)$

$Y(z) = H(z)X(z)$

- System/Transfer function
  - $H(z) = \frac{Y(z)}{X(z)}$

CAUSALITY

- A causal system has $h[n] = 0$ for $n < 0$
  - Right-sided
  - ROC is exterior of circle and includes ∞
- For rational $H(z)$ [diff-eq systems]
  - ROC is outside outermost pole
  - The order of the numerator cannot be greater than the denominator

STABILITY

- The ROC must include the unit circle $|z| = 1$

- For a causal LTI system with rational system function, all poles of $H(z)$ must be inside the unit circle

LTI SYSTEMS FROM DIFFERENCE EQUATIONS

- General difference equation definition

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Take the Z-Transform of both sides

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \quad \Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

  - Always rational
  - Need additional constraints (stable, causal) to determine ROC

EXAMPLE 10.25

- Find the impulse response (assume stable system)

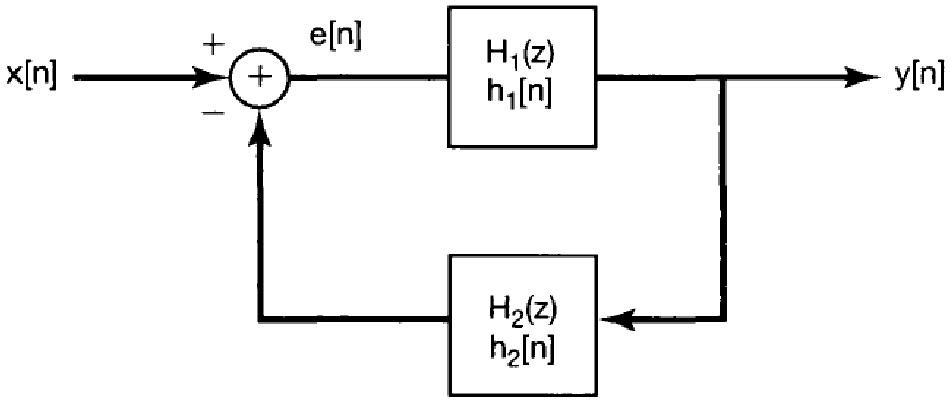
$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATION

CHAPTER 10.8

INTERCONNECTIONS & BLOCK DIAGRAMS

- System functions for interconnections
  - Handled the same as for LT
- Block diagrams
  - Covered in Ch2 notes slide 39
  - Direct Forms: DFI, DFII
  - Cascade Form (factored)
  - Parallel Form (PFE)



- $H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1+H_1(z)H_2(z)}$

EXAMPLE 10.30

- Give forms of
  - $H(z) = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$
- (a) Direct form
- (b) Cascade (factored) form
  - $H(z) = \left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1+\frac{1}{4}z^{-1}}\right)$
- (c) Parallel form (PFE)
  - $H(z) = \frac{2/3}{1+\frac{1}{2}z^{-1}} + \frac{1/3}{1+\frac{1}{4}z^{-1}}$

EXAMPLE 10.30

- Give forms of
  - $H(z) = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$
- (a) Direct form
- (b) Factored form
  - $H(z) = \left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1+\frac{1}{4}z^{-1}}\right)$
- (c) Parallel form (PFE)
  - $H(z) = \frac{2/3}{1+\frac{1}{2}z^{-1}} + \frac{1/3}{1+\frac{1}{4}z^{-1}}$

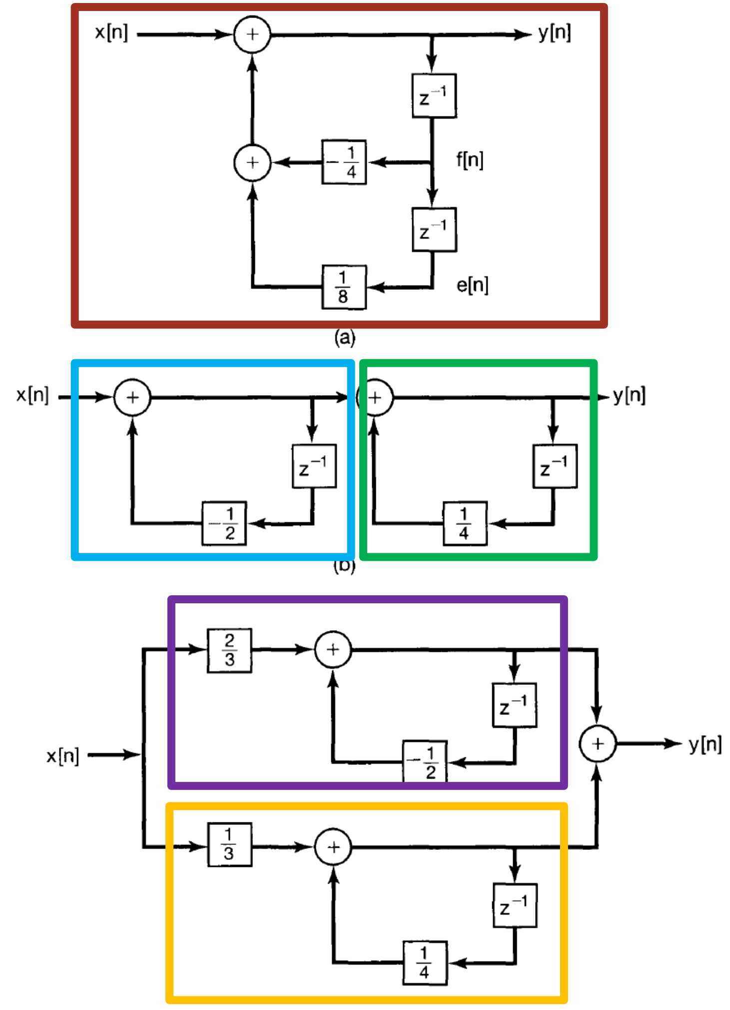


Figure 10.20 Block-diagram representations for the system in Example 10.30: (a) direct form; (b) cascade form; (c) parallel form.

UNILATERAL Z-TRANSFORM

CHAPTER 10.9

UNILATERAL Z-TRANSFORM

- Useful for causal LTI systems with non-zero initial conditions
  - System not initially at rest → system has state/memory

- $X_u(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$
  - Summation only from $[0, \infty]$ while bilateral $[-\infty, \infty]$
  - Results in right-sided sequences (in Z-Transform table)

PROPERTIES OF UNILATERAL (TABLE 10.3)

- Convolution
  - $x_1[n] * x_2[n] \leftrightarrow X_{u1}(z) X_{u2}(z)$
    - $x_1[n] = x_2[n] = 0 \ \forall n < 0$
- Shifting
  - $y[n] = x[n-1] \leftrightarrow Y_u(z) = x[-1] + z^{-1} X_u(z)$
  - Need to generalize for
    - $x[n - n_0] \leftrightarrow ?$
- Example

$$y[n] + 3y[n-1] = x[n]$$

- $H(z) = \frac{1}{1+3z^{-1}}$ (from bilateral)
- Now consider input
  - $x[n] = \alpha u[n] \leftrightarrow X_u(z) = \frac{\alpha}{1-z^{-1}}$

$$\begin{aligned} Y_u(z) &= H(z) X_u(z) \\ &= \left(\frac{1}{1+3z^{-1}}\right)\left(\frac{\alpha}{1-z^{-1}}\right) \\ &= \frac{3/4\alpha}{1+3z^{-1}} + \frac{1/4\alpha}{1-z^{-1}} \end{aligned}$$

- Using unilateral (right-sided) inverse
  - $y[n] = \frac{3}{4}\alpha(-3)^n u[n] + \frac{1}{4}\alpha u[n]$

SOLVING DIFF EQS USING UNILATERAL

- $y[n] + 3y[n-1] = x[n]$, $x[n] = \alpha u[n]$, $y[-1] = \beta$
- $Y_u(z) + 3\{y[-1] + z^{-1}Y_u(z)\} = X_u(z)$
- $Y_u(z)[1 - 3z^{-1}] = \frac{\alpha}{1-z^{-1}} - 3\beta$
- $Y_u(z) = \underbrace{\frac{\alpha}{(1-z^{-1})(1-3z^{-1})}}_{\text{Bilateral solution}} - \underbrace{\frac{3\beta}{1+3z^{-1}}}_{\text{Response to initial conditions}}$

Bilateral solution
Zero initial condition response

Response to initial conditions
Zero-input response

- Can solve each part separately with Z-Transform techniques to find
  - $y[n] = y_{ZICR}[n] + y_{ZIR}[n]$