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EE360: Signals and System I

Schaum's Signals and Systems
Chapter 7
State Space Analysis

http://www.ee.unlv.edu/~b1morris/ee360/

Outline

- Concept of State
- DT State Space Representations
- DT State Equation Solutions
- CT State Space Representations
- CT State Equation Solutions

Introduction to State Space

- So far, we have studied LTI systems based on input-output relationships
  - Known as external description of a system
- Now will examine state space representation of systems
  - Known as internal description of systems

- Consists of two parts
  - State equations – set of equations relating state variables to inputs
  - Output equations – set of equations relating outputs to state variables and inputs

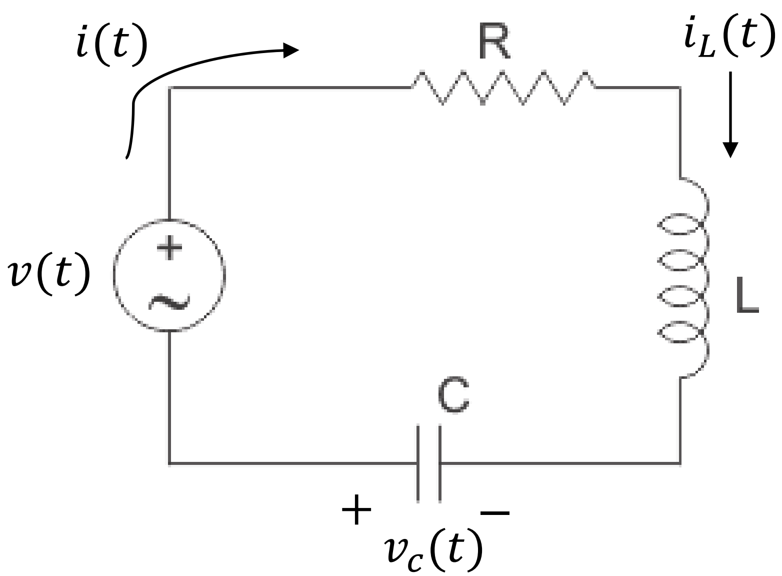
Advantages of State Space

- Provides new insight into system behavior
  - Use of matrix linear algebra
- Can handle multiple-input multiple-output (MIMO) systems
  - Generalize from single-input single-output
- Can be extended to non-linear and time-varying systems
  - General mathematical model
- State equations can be implemented efficiently on computers
  - Enables solving/simulation of complex systems

Definition of State

- State – state of a system @ time t_0 is defined as the *minimal information* that is *sufficient* to determine the state and output of a system for all times $t > t_0$ when the input is also known for $t > t_0$
- State variables (q_i) - variables that contain all state information (memory)
- Note: definition only applies to causal systems

Motivation Example



- Input: $v(t)$
- Output: $i(t)$
- State:
  - $v_L(t) = L\frac{di}{dt}$
  - $i_c(t) = C\frac{dv_c}{dt}$

- Knowing $x(t) = v(t)$ over $[-\infty, t]$ is sufficient to determine $y(t) = i(t)$ over the same interval
- If $x(t)$ is only known between $[t_0, t]$ then the output cannot be determined without knowledge of
  - Current through inductor
  - Voltage across capacitor
- Imagine being handed a circuit (system) that was in operation at time t_0
  - Initial condition problem

Selection of State Variables

- Need to determine "memory elements" of a system

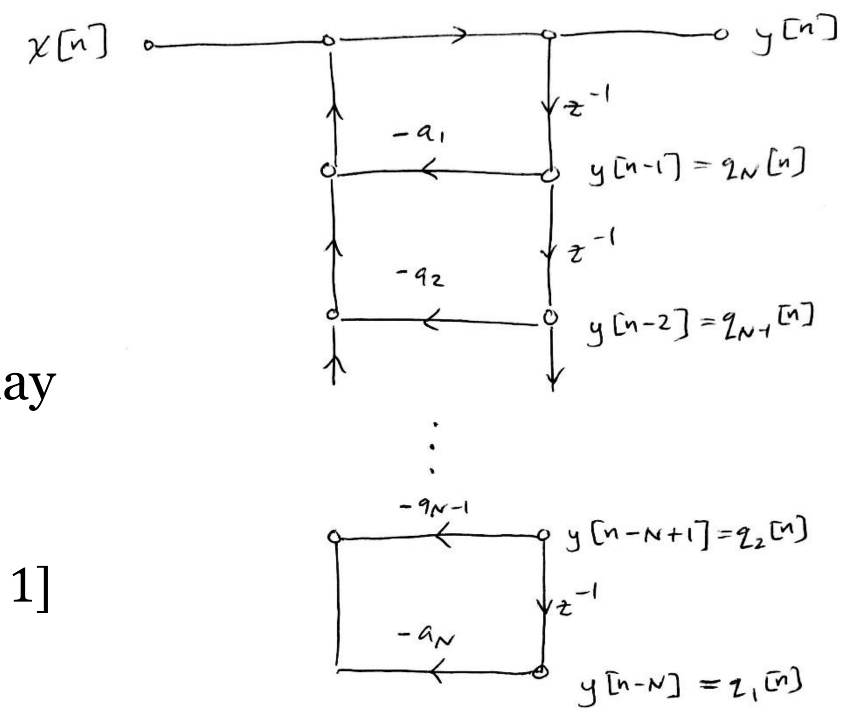
- DT: Select outputs of delay elements
- CT: Select outputs of integrators or energy-storing elements (capacitors, inductors)

- However, state-variable choice is not unique
  - Transformations of variables will result in same state space analysis

DT State Space Representation I

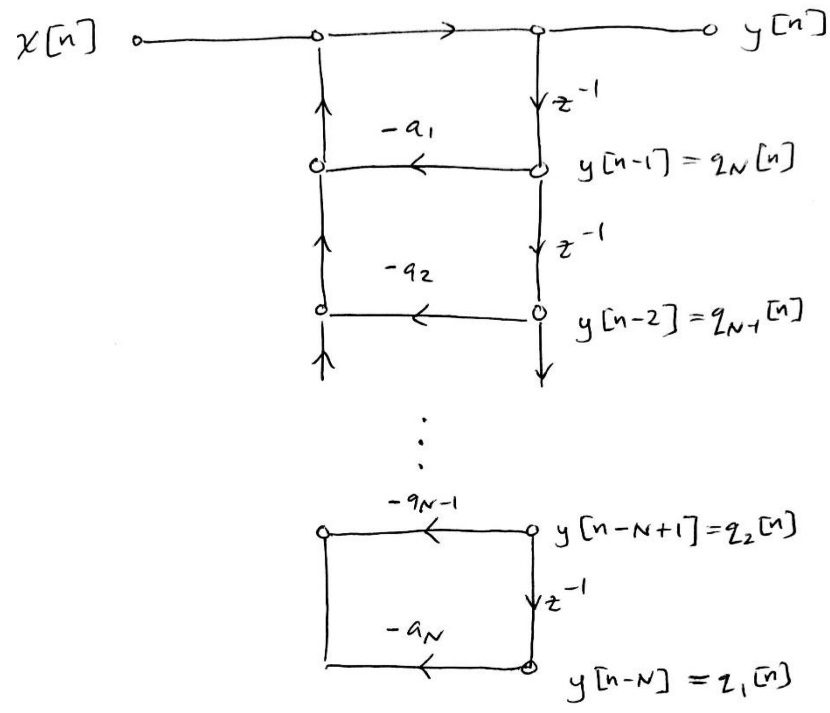
- Consider a single-input single-output (SISO) DT LTI system
  - $y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = x[n]$
- To uniquely determine a complete solution (output), requires N initial conditions
  - $y[-1], y[-2], \dots, y[-N]$
- Define state variables (outputs of delay elements)
  - $q_1[n] = y[n-N]$
  - $q_2[n] = y[n-(N-1)] = y[n-N+1]$
  - …
  - $q_N[n] = y[n-1]$

N state vars



DT State Space Representation II

- Find next (step-ahead) state
  - By definition of delay or using signal flow graph



- $q_1[n+1] = y[n+1-N] = q_2[N]$
- $q_2[n+1] = y[n+1-N+1] = y[n-N+2] = q_3[n]$
- …
- $q_N[n+1] = y[n+1-1] = y[n] = -a_1 y[n-1] + \cdots + -a_N y[n-N]$ (recursive form)

- $q_N[n+1] = -a_1 q_N[n] - a_2 q_{N-1}[n] + \cdots + -a_N q_1[n]$

DT State Space Representation III

- These relationships can be compactly expressed in matrix form

$$\underbrace{\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \\ \vdots \\ q_N[n+1] \end{bmatrix}}_{\underline{q}[n+1]} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ & & \dots & & & \\ -a_N & -a_{N-1} & -a_{N-2} & -a_{N-3} & \dots & -a_1 \end{bmatrix}}_{\substack{\mathbf{A} \\ \text{system matrix}}} \underbrace{\begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_N[n] \end{bmatrix}}_{\underline{q}[n]} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{B}} x[n]$$

$$y[n] = \underbrace{\begin{bmatrix} -a_N & -a_{N-1} & \dots & -a_1 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_N[n] \end{bmatrix}}_{\underline{q}[n]} + \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{\mathbf{D}} x[n]$$

- State equation – next state from past state and input

$$\underline{q}[n+1] = \mathbf{A}\underline{q}[n] + \mathbf{B}\underline{x}[n]$$

- Output equation – output based on state and input

$$\underline{y}[n] = \mathbf{C}\underline{q}[n] + \mathbf{D}\underline{x}[n]$$

- Note: generalized form for MIMO systems with vector $\underline{x}[n], \underline{y}[n]$

DT State Space Representation IV

- Previous example:
  - Defined state variables as outputs of delay elements
  - Rewrote state relationships using a vectorized form of state $\underline{q}[n]$
- Goal: build state-equations given either a difference equation or block-diagram
- Note: previous example had no delayed input $x[n]$. How would delayed inputs change state space representation?
  - Consider DFII structure and develop state equations

Similarity Transformation

- Choice of state-variable is not unique

- Can have another choice of state variables as a transformation
- If $\underline{v}[n] = T\underline{q}[n]$
  - T is $N \times N$ non-singular transformation matrix
- Then, $\underline{q}[n] = T^{-1}\underline{v}[n]$

- You and your friend could have different (valid) state variable choices for same state space representation

Solution to DT State Equations

- Two approaches
  - Time-domain solution
  - Z-transform solution

DT: Time-Domain Solution

$$\underline{q}[n+1] = \mathbf{A}\underline{q}[n] + \mathbf{B}\underline{x}[n]$$
$$\underline{y}[n] = \mathbf{C}\underline{q}[n] + \mathbf{D}\underline{x}[n]$$

- Solve for state iteratively given an initial state $\underline{q}[0]$

$$\begin{aligned}
\underline{q}[n+1] &= \mathbf{A}\underline{q}[n] + \mathbf{B}x[n] \\
\underline{q}[1] &= \mathbf{A}\underline{q}[0] + \mathbf{B}x[0] \\
\underline{q}[2] &= \mathbf{A}\underline{q}[1] + \mathbf{B}x[1] = \mathbf{A}\left\{\mathbf{A}\underline{q}[0] + \mathbf{B}x[0]\right\} + \mathbf{B}x[1] \\
&= \mathbf{A}^2\underline{q}[0] + \mathbf{A}\mathbf{B}x[0] + \mathbf{B}x[1] \\
&\vdots \\
\underline{q}[n] &= \mathbf{A}^n\underline{q}[0] + \mathbf{A}^{n-1}\mathbf{B}x[0] + \ldots + \mathbf{B}x[n-1] \\
&= \mathbf{A}^n\underline{q}[0] + \sum_{k=0}^{n-1}\mathbf{A}^{n-1-k}\mathbf{B}x[k] \qquad n > 0
\end{aligned}$$

- Use this to solve for the output

$$\begin{aligned}
y[n] &= \mathbf{C}\underline{q}[n] + \mathbf{D}x[n] \\
&= \underbrace{\mathbf{C}\mathbf{A}^n\underline{q}[0]}_{\text{zero-input response}} + \underbrace{\sum_{k=0}^{n-1}\mathbf{C}\mathbf{A}^{n-1-k}\mathbf{B}x[k] + \mathbf{D}x[n]}_{\text{zero-state response}}
\end{aligned}$$

DT Z-Transform Solution

- Must use unilateral z-transform due to initial conditions

$$\begin{aligned} \underline{q}[n+1] &= \mathbf{A}\underline{q}[n] + \mathbf{B}\underline{x}[n] \\ \underline{y}[n] &= \mathbf{C}\underline{q}[n] + \mathbf{D}\underline{x}[n] \end{aligned} \iff \begin{aligned} z\underline{Q}_u(z) - z\underline{q}[0] &= \mathbf{A}\underline{Q}_u(z) + \mathbf{B}\underline{X}_u(z) \\ \underline{Y}_u(z) &= \mathbf{C}\underline{Q}_u(z) + \mathbf{D}\underline{X}_u(z) \end{aligned}$$

$$\underline{Q}_u(z) = \begin{bmatrix} Q_{u1}(z) \\ Q_{u2}(z) \\ \vdots \\ Q_{uN}(z) \end{bmatrix}$$

- Rearranging state equation

$$\begin{aligned} zQ_u(z) - AQ_u(z) &= z\underline{q}[0] + BX(z) \\ (zI - A)Q_u(z) &= z\underline{q}[0] + BX(z) \end{aligned}$$

$$Q_u(z) = \underbrace{(zI - A)^{-1} z\underline{q}[0]}_{\text{zero-input}} + \underbrace{(zI - A)^{-1} BX_u(z)}_{\text{zero-state}}$$

$$Zu \updownarrow$$

$$\underline{q}[n] = Z_u^{-1}\left\{(zI - A)^{-1} z\right\}\underline{q}[0] + Z_u^{-1}\left\{(zI - A)^{-1} BX_u(z)\right\}$$

- Plug in for output $y[n]$

$$y[n] = CZ_u^{-1}\left\{(zI - A)^{-1} z\right\}\underline{q}[0] + CZ_u^{-1}\left\{(zI - A)^{-1} BX_u(z)\right\} + Dx[n]$$

System Function with State Equations

- $H(z)$ is defined for zero initial conditions (initial rest or bilateral Z-transform formulation)
  - E.g. $\underline{q}[0] = 0$

$$Q_u(z) = \underbrace{(zI - A)^{-1} z\underline{q}[0]}_{\text{zero-input}} + \underbrace{(zI - A)^{-1} B X_u(z)}_{\text{zero-state}}$$

$$Q(z) = (zI - A)^{-1} B X(z)$$

- Solve for output

$$\begin{aligned} Y(z) &= CQ(z) + DX(z) \\ &= C(zI - A)^{-1} B X(z) + DX(z) \\ &= \underbrace{\left[C(zI - A)^{-1} B + D\right]}_{H(z)} X(z) \end{aligned}$$

Stability (BIBO)

- Given λ_k eigenvalues of system matrix A
  - $|\lambda_k| < 1 \quad \forall k$
  - λ_k must be distinct

- Note: when Schaum's asks about stability they are usually talking about *asymptotically stable* ($|\lambda_k| < 1$)

DT Example Problems

In WebEx lecture

- Problem 7.23
- Problem 7.8

CT State Space Representation I

- Consider a single-input single-output (SISO) CT LTI system
  - $\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_N y(t) = x(t)$
- To uniquely determine a complete solution (output), requires N initial conditions – one set:
  - $y(0), y^{(1)}(0), \ldots, y^{(N-1)}(0)$; where $y^{(k)}(t) = \frac{d^k y(t)}{dt}$
- Define state variables (less obvious for CT)
  - Generally, output of integral block,
  - Here shortcut to $y(t)$ derivatives because no $x(t)$ derivatives

N state vars:

- $q_1(t) = y(t)$
- $q_2(t) = y^{(1)}(t)$
- …
- $q_N(t) = y^{(N-1)}(t)$

CT State Space Representation II

- Find state dot derivative (derivative "feeds" an integral block)
  - $\dot{q}_k(t) = \frac{d}{dt} q_k(t)$

$$\begin{aligned} q_1(t) &= y(t) \\ q_2(t) &= y^{(1)}(t) \\ &\vdots \\ q_N(t) &= y^{(N-1)}(t) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{q}_1(t) &= q_2(t) \\ \dot{q}_2(t) &= q_3(t) \\ &\vdots \\ \dot{q}_N(t) &= -a_N q_1(t) - a_{N-1} q_2(t) - \ldots - a_1 q_N(t) + x(t) \end{aligned}$$

  - Note: $\frac{d^N y(t)}{dt^N} = -a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} - \cdots - a_N y(t) + x(t)$

CT State Space Representation III

- These relationships can be compactly expressed in matrix form

$$\underbrace{\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \vdots \\ \dot{q}_N(t) \end{bmatrix}}_{\underline{\dot{q}}(t)_{N\times 1}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ & & \dots & & & \\ -a_N & -a_{N-1} & -a_{N-2} & -a_{N-3} & \dots & -a_1 \end{bmatrix}}_{\substack{\mathbf{A}_{N\times N} \\ \text{system matrix}}} \underbrace{\begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{bmatrix}}_{\underline{q}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{B}_{N\times 1}} x(t)$$

$$y(t)_{1\times 1} = \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}}_{\mathbf{C}_{1\times N}} \underbrace{\begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{bmatrix}}_{\underline{q}(n)} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}_{1\times 1}} x(t)$$

CT State Space Representation IV

- Generalizes for MIMO systems as

$$\underset{N\times 1}{\underline{\dot{q}}(t)} = \underset{N\times N}{\mathbf{A}}\,\underset{N\times 1}{\underline{q}(t)} + \underset{N\times m}{\mathbf{B}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{bmatrix}}_{\underline{x}(t)_{m\times 1}}$$

$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}}_{\underline{y}(t)_{p\times 1}} = \underset{p\times N}{\mathbf{C}}\,\underset{N\times 1}{\underline{q}(t)} + \underset{p\times m}{\mathbf{D}}\,\underline{x}(t)$$

CT Laplace Transform Solution

- Must use unilateral LT due to initial conditions

$$\begin{aligned} \dot{\underline{q}}(t) &= \mathbf{A}\underline{q}(t) + \mathbf{B}\underline{x}(t) \\ \underline{y}(t) &= \mathbf{C}\underline{q}(t) + \mathbf{D}\underline{x}(t) \end{aligned} \iff \begin{aligned} s\underline{Q}_u(s) - \underline{q}(0) &= \mathbf{A}\underline{Q}_u(s) + \mathbf{B}\underline{X}_u(s) \\ \underline{Y}_u(s) &= \mathbf{C}\underline{Q}_u(s) + \mathbf{D}\underline{X}_u(s) \end{aligned}$$

- Rearranging state equation

$$\begin{aligned} sQ_u(s) - AQ_u(s) &= \underline{q}(0) + BX_u(s) \\ (sI - A)Q_u(s) &= \underline{q}(0) + BX_u(s) \\ \Rightarrow Q_u(s) &= (sI - A)^{-1}\underline{q}(0) + (sI - A)^{-1}BX_u(s) \end{aligned}$$

- Plug in for output

$$\begin{aligned} Y(s) &= C\left[(sI - A)^{-1}\underline{q}(0)\right] + C(sI - A)^{-1}BX_u(s) + DX_u(s) \\ &= \underbrace{C(sI - A)^{-1}\underline{q}(0)}_{\text{zero-input response}} + \underbrace{\left[C(sI - A)^{-1}B + D\right]X_u(s)}_{\text{zero-state response}} \end{aligned}$$

$$L_u \updownarrow$$

$$y(t) = y_{zir}(t) + y_{zsr}(t)$$

Determining System Function

- From previous example
  - $H(s) = \frac{Y(s)}{X(s)} = \underline{c}(sI - A)^{-1}\underline{b} + d$

- When MIMO
  - $\underset{p\times m}{H(s)} = \underset{p\times N}{C} \underset{N\times N}{(sI - A)^{-1}} \underset{N\times m}{B} + \underset{p\times m}{D}$
  - Each element $H_{ij}(s)$ of $H(s)$ matrix is the transfer function relating output $y_i(t)$ to input $x_j(t)$

Stability (BIBO)

- Given λ_k eigenvalues of system matrix A
  1. $Re\{\lambda_k\} < 0 \quad \forall k$
  2. λ_k must be distinct

- Note: when Schaum's asks about stability they are usually talking about *asymptotically stable* $(Re\{\lambda_k\} < 0)$

CT Example Problems

In WebEx lecture

- Problem 7.48