Chapter 3: Fourier Series

Example 3.7

Sometimes is may be easier to calculate the FS of a transformed signal $g(t) = f(x(t))$ rather than $x(t)$ directly. ($f(.)$ function operation). For example, $b_k \longleftrightarrow g(t) = \frac{dx(t)}{dt}$, with $f = \frac{d}{dt}$.

You are told that $x(t)$ has fundamental period T and FS a_k. Given

$$\int_T^{2T} x(t)dt = 2,$$

find a_k in terms of b_k and T.

Solution

$$\int_T^{2T} x(t)dt = 2 = a_0 T \qquad \text{Since,} \quad a_0 = \frac{1}{T}\int_T x(t)e^{-j(0)\omega_0 t}dt$$

$$\Rightarrow a_0 = \frac{2}{T}.$$

From Table 3.1 (pg 206)

$$b_k \longleftrightarrow jk\frac{2\pi}{T}a_k$$

$$\Rightarrow a_k = \frac{b_k}{jk\frac{2\pi}{T}}$$

$$= \begin{cases} \frac{2}{T} & k = 0 \\ \frac{b_k}{jk2\pi/T} & k \neq 0 \end{cases}$$

Example: FS Derivative

Find the FS of the periodic sawtooth wave using technique from previous example. By taking the

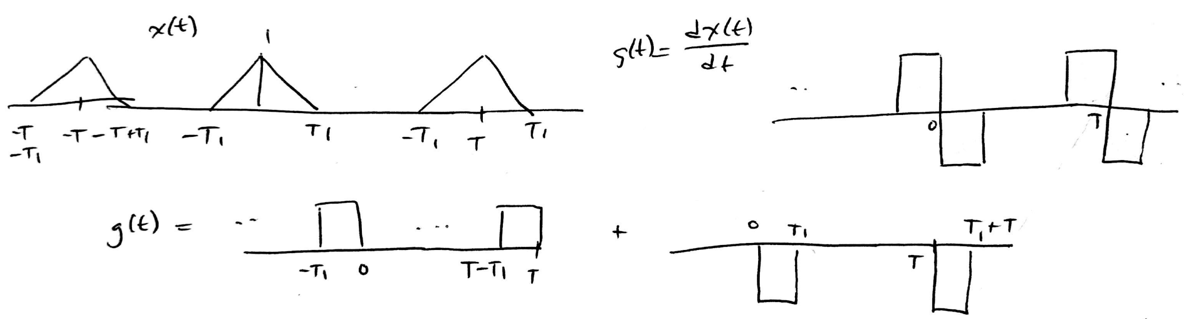


Figure 1: Sawtooth Wave: Decomposition with derivative into the sum of two shifted rectangular pulse trains.

derivative of the sawtooth, you are left with the sum of shifted rectangular pulse trains (known sinc coefficients). The final coefficients b_k are the sum of the two shift sinc coefficients from the rectangular waves.

See examples 3.6 and 3.7 for similar treatment.

3.8 Filtering

Filtering is an important process in many applications
– The goal is to change the relative amplitudes of frequency components in a signal

LTI systems are known as **frequency-shaping filters**
– Those that pass some frequencies and eliminate others are called **frequency-selective filters**
– Common frequency-selective filters include low-pass, high-pass, bandpass, and bandstop (notch)

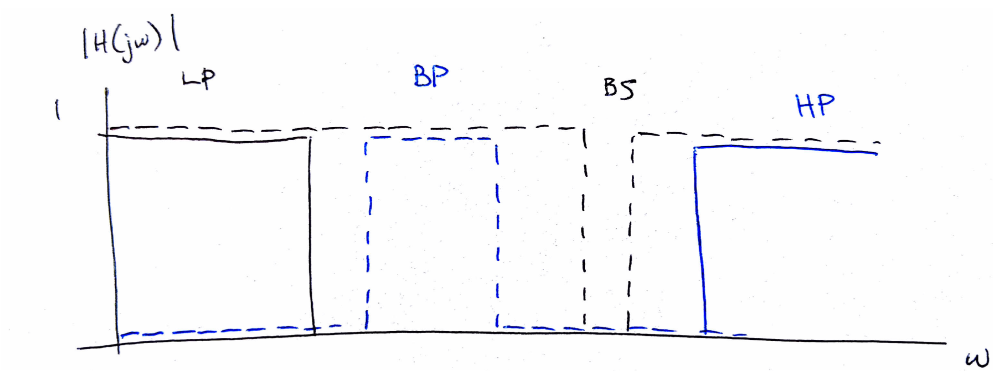


Figure 2: Frequency selective filters. Note, this only shows positive frequency. The filters should be symmetric across the y-axis.

Filtering is accomplished through LTI systems by design of frequency response.

Motivation: Audio Equalizer

An audio equalizer gives a user the ability to adjust the sound quality from a stereo. Basic equalizers (e.g. in a car stereo) allow you to adjust the bass (low frequencies) and treble (high frequencies) to match your taste. Below is an example audio equalizer system curve. Notice the equalizer is

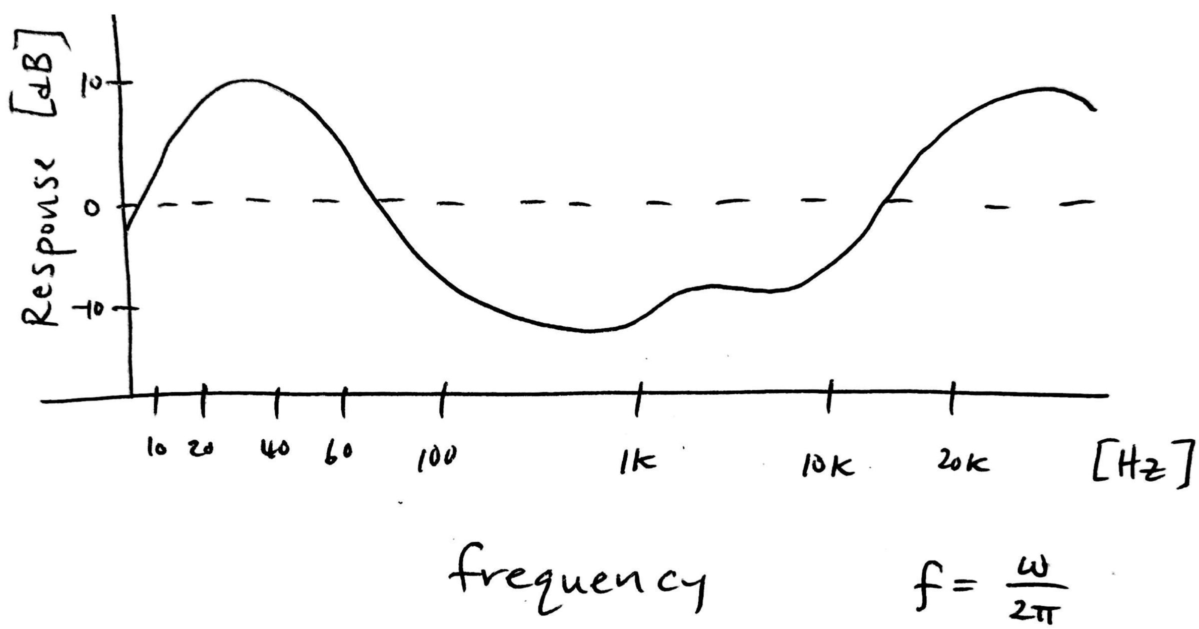


Figure 3: Frequency shaping from an audio equalizer (like in a car stereo or boombox).

plotted using a log-log plot to show a large range of frequencies and response. The response is often represented in decibels

$$\text{dB} = 20\log_{10}|H(j\omega)|$$

where $|H(j\omega)|$, the magnitude response, is most useful for frequency-selective filters.

This particular equalizer is designed to boost low and high frequencies but attenuate mid-frequencies.

Example: Derivative Filters

Characterize the system by plotting magnitude and phase responses:

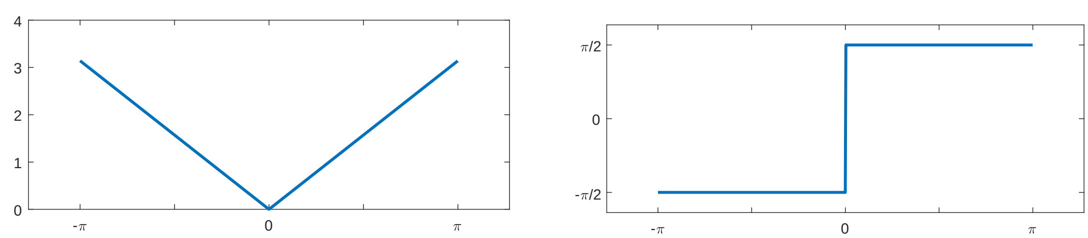
$$y(t) = \frac{d}{dt}x(t).$$

Solution

We'll learn better ways to solve this in the future (Chapter 9). For now we can make use of the eigensignal property of LTI systems.

$$\begin{aligned} x(t) = e^{j\omega t} \longleftrightarrow y(t) &= \frac{d}{dt}x(t) \\ &= j\omega e^{j\omega t} \\ &= H(j\omega)e^{j\omega t} = H(j\omega)x(t) \qquad \text{by eigen property} \\ \Rightarrow H(j\omega) = j\omega & \end{aligned}$$

See below for the magnitude response $|H(j\omega)| = |\omega|$ and phase response $\angle H(j\omega) = \tan^{-1}\left(\frac{Im}{Re}\right)$. Note: the derivative is looking for/responding to large changes in the input signal. This associated system (filter) has a high response for high frequency components (more rapid changing has more response). This type of system is useful for "edge" detection in images for example.



(a) $|H(e^{j\omega})| = |\omega|$ (b) $\angle H(e^{j\omega}) = \tan^{-1}$

Figure 4: Derivative Filter Response

Example: Average Filter

Characterize the system by plotting magnitude and phase responses:

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right) \qquad \text{average of last two samples.}$$

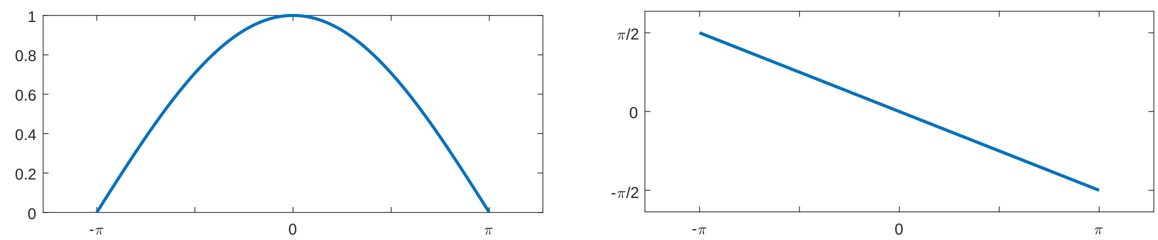
Solution

Again, we'll learn better ways to solve this in the future (Chapter 10). For now we can find

$h[n]$ by having the input be a impulse ($x[n] = \delta[n]$) and taking the Fourier Transform ($H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[n]e^{-j\omega n}$).

$$h[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right) \qquad \longleftrightarrow H(e^{j\omega}) = \frac{1}{2}\left[1 + e^{-j\omega}\right]$$

$$\underbrace{\cos\left(\frac{\omega}{2}\right)}_{|H(e^{j\omega})|}\ \underbrace{e^{-j\omega/2}}_{\angle H(e^{j\omega})}$$



(a) $|H(e^{j\omega})| = \cos(\omega/2)$ (b) $\angle H(e^{j\omega}) = -\omega/2$

Figure 5: Average Filter Response

Matlab for Filters

```
w = -pi:0.01:pi;          %define freq range
H = cos(w/2) .* exp(-j*(w/2));
figure, plot(w, abs(H))
figure, plot(w, phase(H))        %or use angle(H)
```

Final Remarks

Recap of FS

CT DT

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \tag{1}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n} \tag{2}$$

fund. freq ω_0 | fund. freq ω_0 [only 2π freq range]

fund. period $T = \frac{2\pi}{\omega_0}$ | fund. period $N = \frac{2\pi}{\omega_0}$

Further Reading

See Sections 3.10 and 3.11 for more examples of LTI system
– CT systems specified by differential equations
– DT systems specified by difference equations