EE360: Signals and Systems
Z-Transform

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Outline

Introduction

Previously we saw the Laplace Transform

- Extension of FS $\longrightarrow$ FT $\longrightarrow$ LT
- Allowed us to study a wide class of signals/systems (unstable systems with ROC)

The Z-Transform is the discrete version

- While very similar, must recognize the specific differences

Eigensignal Background

Remember

$$x[n] = z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \underbrace{H(z)}_{\text{eigenvalue}} z^n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

Z-Transform Definition

The eigensignal result leads to the definition of the z-Transform

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Shorthand notation

$$x[n] \overset{Z}{\longleftrightarrow} X(z)$$

Fourier Transform Connection

Previously in Laplace, we saw the LT reduced to the FT along the $j\omega$-axis (stability constraint)

For Z-Transform, reduce to the FT along the $e^{j\omega}$ unit circle.

When $z = e^{j\omega}$

$$X(z) =|_{z=e^{j\omega}} = X(e^{j\omega}) = \mathcal{F}\left\{x[n]\right\}$$

where $\mathcal{F}\{.\}$ is the Fourier Transform.

Example 10.1

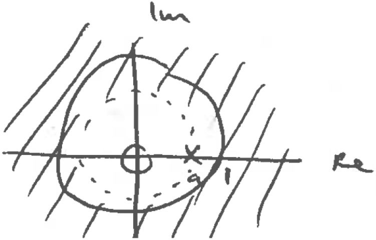
Find Z-transform of input $x[n] = a^n u[n]$

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} \\
&= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \sum_{n=0}^{\infty} \alpha^n \\
&= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}
\end{aligned}
$$

Note: for sum convergence

$$|\alpha| < 1 \quad \Rightarrow |az^{-1}| < 1$$

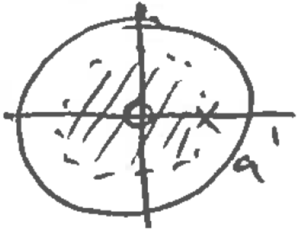
$$\text{ROC: } |z| > |a|$$



Example 10.2

Find Z-transform of input $x[n] = -a^n u[-n-1]$

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=-1}^{\infty} a^{-n} z^n \\
&= 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n \\
&= 1 - \frac{1}{1 - a^{-1} z} = \frac{a^{-1} z}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}}
\end{aligned}
$$



Note: for sum convergence

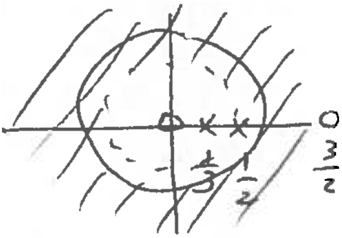
$$
\begin{aligned}
|a^{-1} z| &< 1 \\
\text{ROC: } |z| &< |a|
\end{aligned}
$$

Example 10.3

Find Z-transform of input $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$

Transform each subsignal separately and combine

$$
\begin{aligned}
X(z) &= \underbrace{\frac{7}{1-\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} - \underbrace{\frac{6}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} \\
&= \frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\
&= \frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}
\end{aligned}
$$



ROC must be compatible with $|z| > \frac{1}{3}$ and $|z| > \frac{1}{2}$

$$\text{ROC: } |z| > \frac{1}{2}$$

Example 10.4

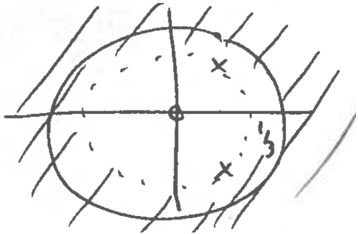
Find Z-transform of input $x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n]$

$$x[n] = \frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n] - \frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n]$$

$$X(z) = \underbrace{\frac{1/2j}{1-\frac{1}{3}e^{j\pi/4}z^{-1}}}_{\substack{|z|>\left|\frac{1}{3}e^{j\pi/4}\right| \\ |z|>\frac{1}{3}}} - \underbrace{\frac{1/2j}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}}}_{\substack{|z|>\left|\frac{1}{3}e^{-j\pi/4}\right| \\ |z|>\frac{1}{3}}}$$

$$= \frac{\frac{1}{3\sqrt{2}}z}{\left(z-\frac{1}{3}e^{j\pi/4}\right)\left(z-\frac{1}{3}e^{-j\pi/4}\right)}$$

$$\text{ROC: } |z| > \frac{1}{3}$$



$\Rightarrow$ For real signals, roots come in complex conjugate pairs

Rational $X(z)$

When $X(z)$ is a ratio of polynomials (or difference equations)

- Pole @ ∞ when the degree of numerator exceeds denominator
- Zero @ ∞ when numerator is of smaller degree than the denominator

Must have balance (equal number of poles and zeros)

9 ROC properties I

1. ROC consists of rings in the z-plane centered about origin
2. ROC does not contain any poles
3. $x[n]$ finite duration, ROC is entire z-plane
   - Except possibly $z = 0$ and/or $z = \infty$ (poles @ zero and/or ∞)
4. $x[n]$ right-sided sequence, if circle $|z| = r_0$ is in the ROC

   then all finite values of z for which $|z| > r_0$ will also be in the ROC
5. $x[n]$ left-sided sequence, if circle $|z| = r_0$ is in the ROC

   then $0 < |z| < r_0$ will also be in the ROC

9 ROC properties II

6. $x[n]$ two-sided, if $|z| = r_0$ is in the ROC, then ROC will be a ring in the z-plane that includes $|z| = r_0$
7. If $X(z)$ is rational, then ROC is bounded by poles or extends to infinity
8. If $X(z)$ is rational and right-sided, then ROC is outside the outermost pole
   - If $x[n]$ is also causal, ROC also includes $z = \infty$
9. If $X(z)$ is rational and left-sided, then ROC is inside innermost pole (not including poles @ $z = 0$)
   - If $x[n]$ is also anticausal ($x[n] = 0\ \forall n > 0$), ROC also includes $z = 0$

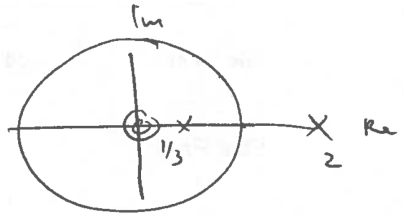
Example 10.8

List all possible ROC for

$$X(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 2z^{-1}\right)} = \frac{z^2}{\left(z - \frac{1}{3}\right)\left(z - 2\right)}$$

Note poles @ $z = \frac{1}{3}$ and $z = 2$.

1. $|z| > 2 \Rightarrow$ right-sided signal
2. $\frac{1}{3} < |z| < 2 \Rightarrow$ two-sided signal
3. $|z| < \frac{1}{3} \Rightarrow$ left-sided signal



Inverse Z-Transform

Definition

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

This is a contour integral within the ROC

Like with LT, will avoid solving this direction and instead use

- inspection method (PFE + known pairs [Table 10.2 pg 776])
- power series expansion

Example 10.10

Find the inverse of

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} \qquad \text{ROC: } \frac{1}{4} < |z| < \frac{1}{3}$$

Do PFE and associate ROCs

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

$$|z| > \frac{1}{4} \qquad |z| < \frac{1}{3}$$

right-sided left-sided

$$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

Power Series Expansion

For finite sequences, can read off $x[n]$ directly by z-power (useful for non-rational z-transform).

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \ldots + x[-1]z^{1} + x[0] + x[1]z^{-1} + \ldots$$

Example 10.12

Find inverse of

$$\begin{aligned} X(z) &= 4z^2 + 2 + 3z^{-1} \qquad 0 < |z| < \infty \\ &= x[-2]z^2 + x[0]z^0 + x[1]z^{-1} \end{aligned}$$

$$\begin{aligned} x[n] &= \begin{cases} 4 & n = -2 \\ 2 & n = 0 \\ 3 & n = 1 \end{cases} \\ &= 4\delta[n+2] + 2\delta[n] + 3\delta[n-1] \end{aligned}$$

Geometric Interpretation of FT

This section will be skipped $\Rightarrow$ you will not be responsible for material

Will save this for EE480 DSP

Properties of Z-Transform I

Same idea as for LT

Will revisit this later

Common Z-Transform Pairs

Will very rarely computer z-transform directly from summation definition.

Bookmark

- Table 10.1 Properties of Z-Transform [page 775]
- Table 10.2 Transform Pairs [page 776]

LTI Systems and Z-Transform

$$\begin{matrix} x[n] \\ X(z) \end{matrix} \longrightarrow \boxed{\mathrm{H(z)}} \longrightarrow \begin{matrix} y[n] \\ Y(z){=}H(z)X(z) \end{matrix}$$

System/Transfer Function

$$H(z) = \frac{Y(z)}{X(z)}$$

Causality

A causal system has $h[n] = 0$ for $n < 0$

- Right-sided
- ROC is exterior of circle and includes ∞

For rational $H(z)$

- ROC is outside outermost pole
- Order of numerator cannot be greater than denominator

Stability

ROC must inlcude the unit circle $|z| = 1$.

For causal LTI system with rational system function, all poles of $H(z)$ must be inside the unit circle

LTI Systems from Difference Equations

General difference equation definition

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Take Z transform of both sides

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Always rational
- Need additional constraints (stable, causal) to determine ROC

Example 10.25

Find the impulse response (assume stable system)

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = X(z)\left[1 + \frac{1}{3}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

Use time-shift property on 2nd term

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

Notice causal by $h[n]$ or ROC including ∞