Professor Brendan Morris, SEB 3216, brendan.morris@unlv.edu

EE360: Signals and System I

Chapter 9
Laplace Transform

http://www.ee.unlv.edu/~b1morris/ee360/

LTI Eigensignals

- Recall:
- $x(t) = e^{st} \rightarrow y(t) = H(s)e^{st}$
- Transfer function: $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$

- Let $s = j\omega$ for Fourier Transform
- $H(s)|_{s=j\omega} = H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$

Laplace Transform

- Define LP for $s = \sigma + j\omega \in \mathbb{C}$
- $X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$

- Notation:
- $X(s) = \mathcal{L}\{x(t)\}$
- $x(t) \leftrightarrow \mathrm{X(s)}$

Laplace Transform Tables

- Last time, we computed the LT directly from the integral definition
  - Learned that there is an algebraic form for LT as well as the region of convergence (ROC)
  - Both are needed to uniquely identify the LT

- More often, we will use tables of known transform pairs (Table 9.2) and transform properties (Table 9.1).

ROC for Laplace Transform

- Algebraic expressions of LT are not sufficient for distinguishing LT, must also include ROC

- 8 properties of ROC
- 1: ROC consists of strips parallel to $j\omega$-axis
  (right-half plane, left-half plane, strip)
  ROC only depends on $Re\{s\} = \sigma \Rightarrow$ vertical lines

- 2: Rational $X(s)$ does not contain any poles
  $X(s)$ is infinite at pole $\Rightarrow$ not stable

ROC for Laplace Transform II

- 3: If $x(t)$ is finite duration and absolutely integrable, then ROC is the entire s-place

- 4: If $x(t)$ is right-sided and $\{s|Re\{s\} = \sigma_0\} \subseteq$ ROC, then $\{s|Re\{s\} > \sigma_0\} \subseteq$ ROC
    - Right-sided ROC

- 5: If $x(t)$ is left-sided and $\{s|Re\{s\} = \sigma_0\} \subseteq$ ROC, then $\{s|Re\{s\} < \sigma_0\} \subseteq$ ROC
    - Left-sided ROC

ROC for Laplace Transform III

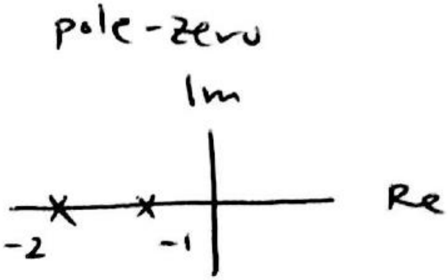
- 6: If $x(t)$ is two-sided (not-bounded) and $\{s|Re\{s\} = \sigma_0\} \subseteq$ ROC, then the ROC is a vertical strip containing the line $Re\{s\} = \sigma_0$

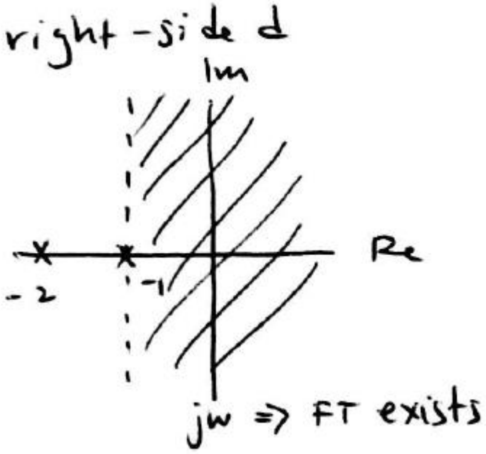
- 7: If the LT $X(s)$ is rational, the ROC is bounded by poles or extends to infinity (pole @ ∞)

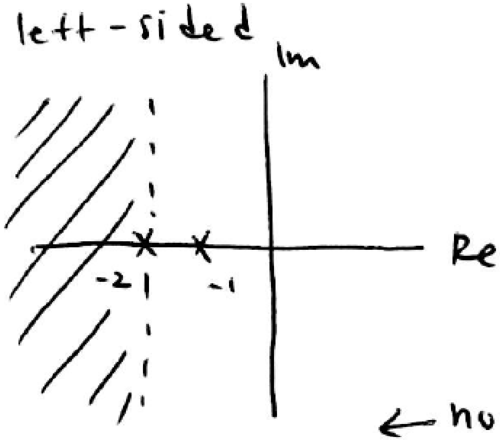
- 8: If $X(s)$ is rational, then
  - $x(t)$ right-sided, ROC is the right-half plane to the right of the right-most pole
  - $x(t)$ left-sided, ROC is the left-half plane to the left of the left-most pole

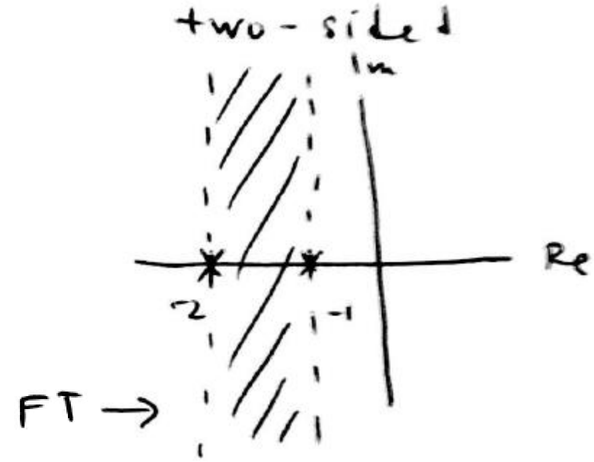
Example ROC Properties

- $X(s) = \frac{1}{(s+1)(s+2)}$









Inverse LT

- By using inverse FT we can find the definition of the inverse LT

$$x(t) = \frac{1}{2\pi} \int_{\sigma - j\omega}^{\sigma + j\omega} X(s) e^{st} ds$$

  - The synthesis equation is a weighted sum of complex exponentials
- Note: this is a contour integral along any vertical line in the ROC of s-plane!
  - We won’t do it this way, instead will rely on PFE techniques in lookup tables

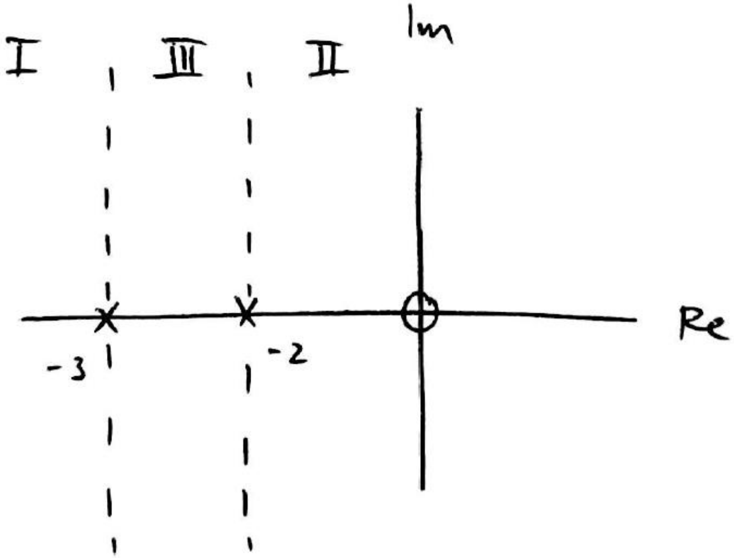
Example Inverse LT

$$X(s) = \frac{s}{s^2 + 5s + 6}$$
$$= \frac{s}{(s+2)(s+3)}$$
$$= \frac{A}{s+2} + \frac{B}{s+3}$$

- Solve for partial terms
- $A = \left.\frac{s}{s+3}\right|_{s=-2} = -2$
- $B = \left.\frac{s}{s+2}\right|_{s=-3} = 3$
- Combine for final answer

$$X(s) = -\frac{2}{s+2} + \frac{3}{s+3}$$

- What are the possible ROC's?



Example Inverse LT II

- ROC I
  - $Re\{s\} < -3 \quad \Longrightarrow$ left-sided signal
  - $X(s) = -\frac{2}{s+2} + \frac{3}{s+3} \leftrightarrow x(t) =$ $-\,2[-e^{-2t}u(-t)] + 3[-e^{-3t}u(-t)]$
  - Notice: the use of left-sided transform pair

- ROC II
  - $Re\{s\} > -2 \quad \Longrightarrow$ right-sided signal
  - $X(s) \leftrightarrow x(t) = -2[e^{-2t}u(t)] + 3[e^{-3t}u(t)]$

Example Inverse LT III

- ROC III
  - $-3 < Re\{s\} < -2 \implies$ two-sided signal
  - Want intersection of
    - Left-sided $\frac{1}{s+2}$ term $(Re\{s\} < -2)$
    - Right-sided $\frac{1}{s+3}$ term $(Re\{s\} > -3)$
  - $X(s) \leftrightarrow x(t) = -2[-\underset{\underset{\text{Left-sided}}{\uparrow}}{e^{-2t}}u(-t)] + 3[\underset{\underset{\text{Right-sided}}{\uparrow}}{e^{-3t}}u(t)]$

Properties of LT (Table 9.1)

- Linearity
  - $x_1(t) \leftrightarrow X_1(s), \mathrm{ROC} = R_1$
  - $x_2(t) \leftrightarrow X_2(s), \mathrm{ROC} = R_2$
- $ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)$
  - $\mathrm{ROC} \supset R_1 \cap R_2$
  - Contains at the very least intersection but could be more

- Linearity Example

$x(t) = x_1(t) - x_2(t)$

$X_1(s) = \frac{1}{s+1}$

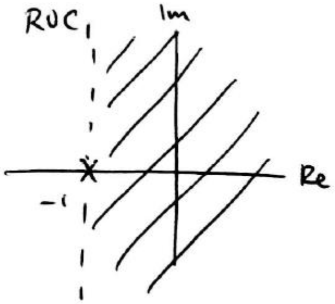
ROC$_1$ $Re\{s\} > -1$

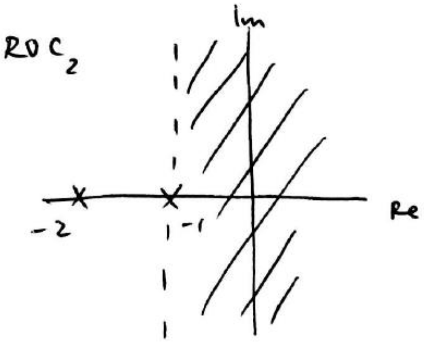
$X_2(s) = \frac{1}{(s+1)(s+2)}$

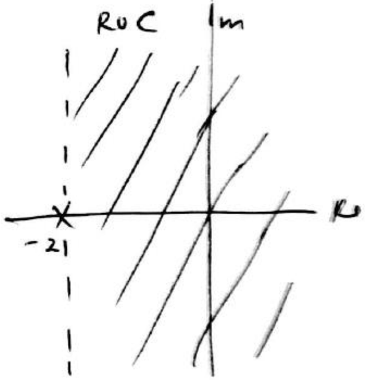
ROC$_2$ $Re\{s\} > -1$

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+2)(s+1)}$$

$$= \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$ due to pole-zero cancellation







Note: ROC larger than ROC$_1$ $\cap$ ROC$_2$ because zero and pole cancel @ $s = -1$

Properties of LT II

- Time-shift
  - $x(t) \longleftrightarrow X(s), \mathrm{ROC} = R$
  - $x(t - t_0) \longleftrightarrow e^{-st_0}X(s), \mathrm{ROC} = R$
- S-domain shift
  - $e^{s_0 t}x(t) \longleftrightarrow X(s - s_0), \mathrm{ROC} = R + Re\{s_0\}$

- Convolution property
  - $x_1(t) * x_2(t) \longleftrightarrow X_1(s)X_2(s), \mathrm{ROC} \supset \mathrm{R}_1 \cap R_2$
  - Key for LTI systems described by differential equations

Properties of LT III

- Differentiation in time
  - $\frac{dx(t)}{dt} \longleftrightarrow sX(s)$, ROC $\supset R$
  - More generally,
  - $\frac{d^k}{dt^k} x(t) \longleftrightarrow s^k X(s)$, ROC $\supset R$
- Integration in time
  - $\int_{-\infty}^{t} x(\tau) d\tau \longleftrightarrow \frac{1}{s} X(s)$, ROC $\supset R \cap \{s | Re\{s\} > 0\}$
    - With $x(t) = \delta(t)$,
    - $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \longleftrightarrow \frac{1}{s}$, ROC $= \{s | Re\{s\} > 0\}$
  - Integration is useful for block diagram representations

LTI Systems and LT

- Reminder,

- Convolution
- $y(t) = x(t) * h(t) \longleftrightarrow Y(s) = X(s)H(s)$
  - $H(s)$ is the system function or transfer function

- Eigensignal
- $x(t) = e^{st} \longleftrightarrow y(t) = H(s)e^{st}$
  - For s in the ROC of $H(s)$
  - If $s = j\omega$ is in the ROC, then the FT exists and $H(j\omega)$ is the frequency response

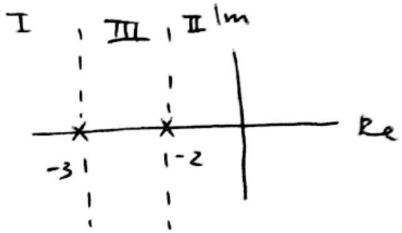
LTI System Properties from $H(s)$

- Many properties of LTI system can be determined directly from system function $H(s)$
  - 1: Causality – a causal system has an ROC that is a right half plane
  - 2: Stability – a system is stable iff ROC of $H(s)$ contains the entire $j\omega$-axis
- For rational $H(s)$ [ratio of polynomial functions as in diff eqns], more can be specified
  - 1r: Causality – the system is causal iff ROC is the right-half plane to the right of the right-most pole
  - 2r: Stability – a causal system is stable iff all poles lie in the half-plane to the left of the $j\omega$-axis
    - All poles have negative real parts

Example

Example

$$\underset{\substack{\uparrow \\ \text{rational}}}{H(s)} = \frac{s}{(s+2)(s+3)} = \frac{-2}{s+2} + \frac{3}{s+3}$$



ROC I

not causal (anti-causal
not right-half plane)

not stable
no jw-axis)

$$h_I(t) = 2e^{-2t}u(-t) - 3e^{-3t}u(-t)$$

ROC II

causal
right-half plane ,

stable
contains jw-axis ,

$$h_{II}(t) = -2e^{-2t}u(t) + 3e^{-3t}u(t)$$

ROC III

not causal ,

not stable ,

$$h_{III}(t) = 2e^{-2t}u(-t) + 3e^{-3t}u(t)$$

Differential Equation LTI Systems

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- Use differentiation in time-domain property

$$\left[\sum_{K=0}^{N} a_k s^k\right] Y(s) = \left[\sum_{K=0}^{M} b_k s^k\right] X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \qquad \begin{array}{l} \longleftarrow \text{Zeros} \\ \longleftarrow \text{Poles} \end{array}$$

Example

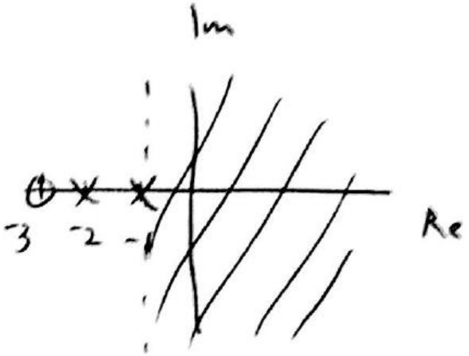
given causal LTI system described by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$$

find $h(t)$

$$Y(s)\left[s^2 + 3s + 2\right] = X(s)\left[s+3\right]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+2)(s+1)} \Rightarrow$$



$$= \frac{A}{s+2} + \frac{B}{s+1} = \frac{-1}{s+2} + \frac{2}{s+1}$$

causality implies

$$ROC = \{s \,/\, Re\{s\} > -1\}$$

$$A = \left.\frac{s+3}{s+1}\right|_{s=-2} = \frac{1}{-1} = -1$$

$$B = \left.\frac{s+3}{s+2}\right|_{s=-1} = \frac{2}{1} = 2$$

$$h(t) = -e^{-2t}u(t) + 2e^{-t}u(t)$$