

Homework #6

Due Mo. 4/08

Note: The **Basic Problems with Answers** will be worth half as much as the other questions. You must show all your work to receive credit.

1. (OW 9.29)

Solution

(a)

$$x(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} \quad \mathcal{Re}\{s\} > -1$$

$$h(t) = e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2} \quad \mathcal{Re}\{s\} > -2$$

(b)

$$Y(s) = X(s)H(s) = \frac{1}{(s+1)(s+2)} \quad \mathcal{Re}\{s\} > -1$$

(c)

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \left. \frac{1}{s+2} \right|_{s=-1} = 1$$

$$B = \left. \frac{1}{s+1} \right|_{s=-2} = -1$$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

(d)

$$y(t) = x(t) * y(t) = \int_0^t e^{-2\tau} e^{-(t-\tau)} d\tau \quad t > 0$$

$$= e^{-t} \int_0^t e^{-\tau} d\tau = e^{-t} [-e^{-\tau}]_0^t = e^{-t} [1 - e^{-t}]$$

$$= [e^{-t} - e^{-2t}]u(t)$$

2. (OW 9.30)

Solution

$$x(t) = u(t) \longleftrightarrow s(t) = (1 - e^{-t} - te^{-t})u(t)$$

$$X(s) = \frac{1}{s} \quad \mathcal{Re}\{s\} > 0$$

$$S(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} \quad \mathcal{Re}\{s\} > 0$$

$$= \frac{(s+1)^2 - s(s+1) - s}{s(s+1)^2} = \frac{1}{s(s+1)^2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(s+1)^2} \quad \mathcal{Re}\{s\} > -1$$

Using the output $y(t) = (2 - 3e^{-t} - e^{-3t})u(t)$

$$\begin{aligned}
 Y(s) &= \frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+3} \quad \mathcal{Re}\{s\} > 0 \\
 &= \frac{2(s+1)(s+3) - 3s(s+3) - s(s+1)}{s(s+1)(s+3)} = \frac{6}{s(s+1)(s+3)} \\
 X(s) &= \frac{Y(s)}{H(s)} = \frac{6(s+1)}{s(s+3)} = 6 \left[\frac{A}{s} + \frac{B}{s+3} \right] \\
 A &= \left. \frac{s+1}{s+3} \right|_{s=0} = \frac{1}{3} \\
 B &= \left. \frac{s+1}{s} \right|_{s=-3} = \frac{-2}{-3} = \frac{2}{3} \\
 X(s) &= \frac{2}{s} + \frac{4}{s+3} \quad \mathcal{Re}\{s\} > 0 \\
 x(t) &= 2u(t) + 4e^{-3t}u(t)
 \end{aligned}$$

3. (OW 9.31)

Solution

(a) See Fig. 1 for the pole plot.

$$\begin{aligned}
 \frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) &= x(t) \\
 [s^2 - s - 2]Y(s) &= X(s)
 \end{aligned}$$

$$\begin{aligned}
 H(s) &= \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \\
 A &= \left. \frac{1}{s+1} \right|_{s=2} = \frac{1}{3} \\
 B &= \left. \frac{1}{s-2} \right|_{s=-1} = \frac{1}{-3} \\
 H(s) &= \frac{1/3}{s-2} - \frac{1/3}{s+1}
 \end{aligned}$$

(b) (i) stable $\Rightarrow -1 < \mathcal{Re}\{s\} < 2$

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)$$

(ii) causal $\Rightarrow \mathcal{Re}\{s\} > 2$

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t)$$

(iii) neither $\Rightarrow \mathcal{Re}\{s\} < -1$

$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t)$$

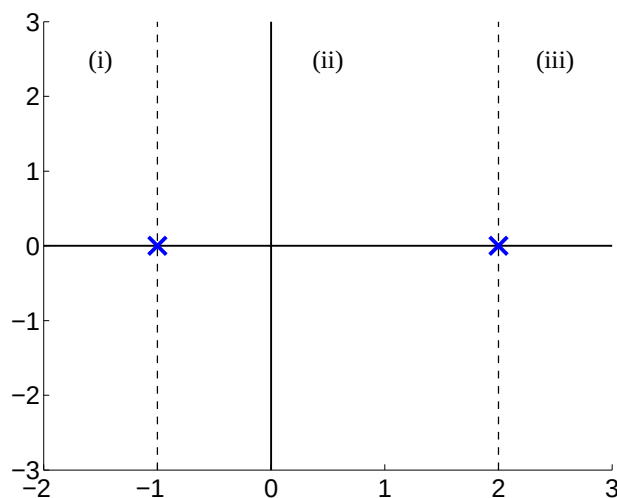


Figure 1: OW 9.31

4. (OW 9.33)

Solution

$$H(s) = \frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1+j)(s+1-j)} \quad \mathcal{Re}\{s\} > -1$$

$$x(t) = e^{-|t|} = e^t u(-t) + e^{-t} u(t)$$

$$X(s) = \frac{-1}{s-1} + \frac{1}{s+1} = \frac{-2}{(s-1)(s+1)} \quad -1 < \mathcal{Re}\{s\} < 1$$

$$Y(s) = X(s)H(s) = \left[\frac{-2}{(s-1)(s+1)} \right] \left[\frac{s+1}{s^2+2s+2} \right] = \frac{-2}{(s-1)(s^2+2s+2)}$$

$$= \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+2}$$

$$A = \left. \frac{-2}{s^2+2s+2} \right|_{s=1} = -\frac{2}{5}$$

$$(Bs+C)(s-1) + A(s^2+2s+2) = -2$$

$$A+B=0 \Rightarrow B = -A = \frac{2}{5}$$

$$-C+2A = -2 \Rightarrow C = 2A+2 = \frac{6}{5}$$

$$Y(s) = -\frac{2/5}{s-1} + \frac{2/5s+6/5}{s^2+2s+2} \quad -1 < \mathcal{Re}\{s\} < 1$$

$$= -\frac{2/5}{s-1} + \frac{2}{5} \left[\frac{s+1}{(s+1)^2+1} \right] + \frac{4}{5} \left[\frac{1}{(s+1)^2+1} \right]$$

$$y(t) = \frac{2}{5} e^t u(-t) + \frac{2}{5} e^{-t} \cos(t) u(t) + \frac{4}{5} e^{-t} \sin(t) u(t)$$

5. (OW 9.40)

Solution

(a)

$$Y(s)[s^3 + 6s^2 + 11s + 6] = X(s) = \frac{1}{s+4}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^3 + 6s^2 + 11s + 6} = \frac{1}{(s+1)(s+2)(s+3)}$$

$$Y(s) = X(s)H(s) = \frac{1}{(s+1)(s+2)(s+3)(s+4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

$$A = \left. \frac{1}{(s+2)(s+3)(s+4)} \right|_{s=-1} = \frac{1}{6}$$

$$B = \left. \frac{1}{(s+1)(s+3)(s+4)} \right|_{s=-2} = -\frac{1}{2}$$

$$C = \left. \frac{1}{(s+1)(s+2)(s+4)} \right|_{s=-3} = \frac{1}{2}$$

$$D = \left. \frac{1}{(s+1)(s+2)(s+3)} \right|_{s=-4} = -\frac{1}{6}$$

$$Y(s) = \frac{1/6}{s+1} - \frac{1/2}{s+2} + \frac{1/2}{s+3} - \frac{1/6}{s+4}$$

Since a causal response is required for the zero-state response (ZSR)

$$y_{zsr}(t) = \left[\frac{1}{6}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-3t} - \frac{1}{6}e^{-4t} \right] u(t)$$

(b) The zero-input response (ZIR) uses the initial conditions and unilateral Laplace Transform with $X_u(s) = 0$.

$$s^3 Y_u(s) - s^2 y(0^-) - s y'(0^-) - y''(0^-) + 6 [s^2 Y_u(s) - s y(0^-) - y'(0^-)] + 11 [s Y_u(s) - y(0^-)] + 6 Y_u(s) = X_u(s)$$

$$Y_u(s) [s^3 + 6s^2 + 11s + 6] -$$

$$[s^2 y(0^-) + s(y'(0^-) + 6y(0^-)) + y''(0^-) + 6y'(0^-) - 11y(0^-)] = 0$$

$$Y_u(s) [s^3 + 6s^2 + 11s + 6] - [s^2 + 5s + 6] = 0$$

$$Y_u(s) [s^3 + 6s^2 + 11s + 6] = s^2 + 5s + 6$$

$$Y_u(s) = \frac{s^2 + 5s + 6}{s^3 + 6s^2 + 11s + 6} = \frac{(s+3)(s+2)}{(s+1)(s+2)(s+3)} = \frac{1}{s+1}$$

$$y_{zir}(t) = e^{-t} u(t)$$

(c) The total response is the sum of the ZSR and ZIR (for input $x(t) = e^{-4t} u(t)$)

$$y(t) = y_{zsr}(t) + y_{zir}(t) = \left[\frac{7}{6}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-3t} - \frac{1}{6}e^{-4t} \right] u(t)$$

6. (OW 9.45)

Solution

(a)

$$\begin{aligned}
 X(s) &= \frac{s+2}{s-2} & \mathcal{R}e\{s\} < 2 \\
 y(t) &= -\frac{2}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(t) \\
 Y(s) &= \frac{2/3}{s-2} + \frac{1/3}{s+1} = \frac{s}{(s+1)(s-2)} & -1 < \mathcal{R}e\{s\} < 2 \\
 H(s) &= \frac{Y(s)}{X(s)} = \frac{s}{(s+1)(s+2)} & \mathcal{R}e\{s\} > -1
 \end{aligned}$$

(b)

$$\begin{aligned}
 H(s) &= \frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \\
 &= A = \left. \frac{s}{s+2} \right|_{s=-1} = -1 \\
 &= B = \left. \frac{s}{s+1} \right|_{s=-2} = 2 \\
 H(s) &= \frac{-1}{s+1} + \frac{2}{s+2} \\
 h(t) &= -e^{-t}u(t) + 2e^{-2t}u(t)
 \end{aligned}$$

(c) Remember e^{st} is an eigen-signal for an LTI system and use $H(s)$ from above to solve.

$$x(t) = e^{3t} \longleftrightarrow y(t) = H(s)|_{s=3} e^{3t} = \frac{3}{20} e^{3t}$$

7. (OW 9.48)

Solution

(a)

$$\begin{aligned}
 \delta(t) &= h(t) * h_1(t) \xleftrightarrow{\mathcal{L}} 1 = H(s)H_1(s) \\
 H_1(s) &= \frac{1}{H(s)}
 \end{aligned}$$

(b) See the pole-zero plot in Fig. 2.

$$H_1(s) = \frac{1}{H(s)} = \frac{s+1}{s-1/2}$$

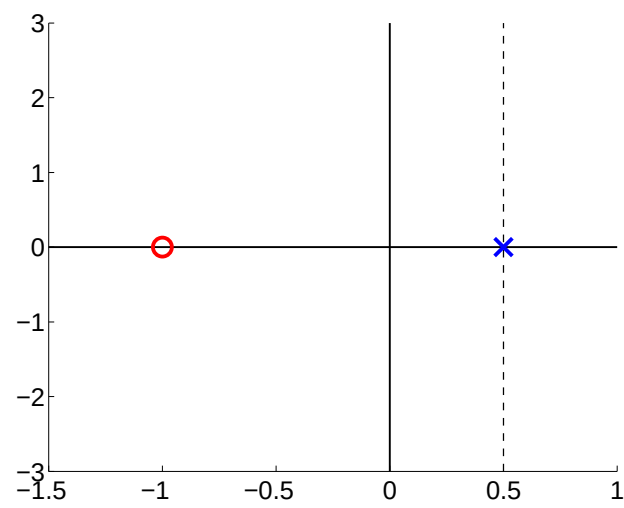


Figure 2: OW 9.48