Homework #4
Due Mo. 3/13

Note: The **Basic Problems with Answers** will be worth half as much as the other questions. You must show all your work to receive credit.

1. (OW 3.21)

   Solution

   Given the following information about the signal

   $$T = 8, \qquad a_1 = a_{-1}^* = j, \qquad a_5 = a_{-5} = 2,$$

   the periodic signal $x(t)$ can be expressed using the complex exponential form of the Fourier Series and converted using $\omega_0 = 2\pi/T = \pi/4$.

   $$\begin{aligned} x(t) &= a_{-5}e^{-j5\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_1 e^{j\omega_0 t} + a_5 e^{j5\omega_0 t} \\ &= 2\left[e^{-j5\omega_0 t} + e^{j5\omega_0 t}\right] + j\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] \\ &= 4\cos(5\omega_0 t) - 2\sin(\omega_0 t) \\ &= 4\cos\left(\frac{5\pi}{4}t\right) - 2\cos\left(\frac{\pi}{4}t - \frac{\pi}{2}\right) \end{aligned}$$

2. (OW 3.24)

   Solution

   (a) Find the average signal value over a period

   $$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{2}\left[\frac{1}{2}(1)(2)\right] = \frac{1}{2}.$$

   (b) Define

   $$g(t) = \frac{dx(t)}{dt} = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 \le t \le 2 \end{cases}.$$

   Then FS coefficients can be found by noting

   $$g(t) \longrightarrow b_k = \frac{1}{T}\int_T g(t)e^{-jk\omega_0 t}dt.$$

   $k = 0$ $\qquad b_0 = 0$ $\qquad$ average over single period

   $k \neq 0$

   $$\begin{aligned} b_k &= \frac{1}{2}\left[\int_0^1 e^{-jk\omega_0 t}dt - \int_1^2 e^{-jk\omega_0 t}dt\right] \\ &= \frac{1}{2}\left[\frac{1}{-jk\omega_0}e^{-jk\omega_0}\right]_0^1 - \frac{1}{2}\left[\frac{1}{-jk\omega_0}e^{-jk\omega_0}\right]_1^2 \\ &= \frac{1}{2jk\omega_0}\left[1 - e^{-j\omega_0}\right] + \frac{1}{2jk\omega_0}\left[e^{-j2\omega_0} - e^{-j\omega_0}\right] \\ &= \frac{1}{2jk\omega_0}\left[1 - 2e^{-jk\omega_0} + e^{-jk2\omega_0}\right] \\ &= \frac{1}{2jk\pi}\left[1 - 2e^{-jk\pi} + \underbrace{e^{-jk2\pi}}_{=1}\right] \\ &= \frac{1}{jk\pi}\left[1 - e^{-jk\pi}\right] \end{aligned}$$

(c) Using the differentiation property

$$
\begin{aligned}
x(t) &\longleftrightarrow a_k \\
\frac{dx(t)}{dt} &\longleftrightarrow b_k = jk\omega_0 a_k = jk\pi a_k \\
&\Rightarrow a_k = \frac{1}{jk\pi} b_k = \frac{1}{(jk\pi)^2}[1 - e^{-j\pi}] = \frac{1}{k^2\pi^2}[e^{-j\pi} - 1]
\end{aligned}
$$

therefore,

$$
a_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{k^2\pi^2}[e^{-j\pi} - 1] & else \end{cases}.
$$

3. (OW 3.25)

Solution

(a)

$$
\begin{aligned}
x(t) &= \frac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] \\
\Rightarrow a_k &\longrightarrow a_1 = a_{-1} = \frac{1}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
y(t) &= \frac{1}{2j}[e^{j4\pi t} - e^{-j4\pi t}] \\
\Rightarrow b_k &\longrightarrow b_1 = \frac{1}{2j}, \quad b_{-1} = -\frac{1}{2j}
\end{aligned}
$$

(c) The multiplicative property states the product in the time domain has convolution of the FS coefficients.

$$
z(t) = x(t)y(t) \longleftrightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.
$$

$$
\begin{aligned}
c_k &= [\frac{1}{2}\delta(k+1) + \frac{1}{2}\delta(k-1)] * [-\frac{1}{2j}\delta(k+1) + \frac{1}{2j}\delta(k-1)] \\
&= -\frac{1}{4j}\delta(k+2) + \frac{1}{4j}\delta(k) - \frac{1}{4j}\delta(k) + \frac{1}{4j}\delta(k-2) \\
&= \underbrace{-\frac{1}{4j}}_{c_{-2}}\delta(k+2) + \underbrace{\frac{1}{4j}}_{c_2}\delta(k-2)
\end{aligned}
$$

(d) Using pure trigonometric expansion

$$
\begin{aligned}
z(t) &= \cos(4\pi t)\sin(4\pi t) \\
&= \frac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] \cdot \frac{1}{2j}[e^{j4\pi t} - e^{-j4\pi t}] \\
&= \frac{1}{4j}[e^{j2(4\pi)t} - e^0 + e^0 - e^{-j2(4\pi)t}] \\
&= \underbrace{\frac{1}{4j}}_{c_2} e^{j2(4\pi)t} \underbrace{-\frac{1}{4j}}_{c_{-2}} e^{-j2(4\pi)t}
\end{aligned}
$$

4. (OW 3.27)

Solution

$$N = 5, \qquad a_0 = 2, \qquad a_2 = a_{-2}^* = 2e^{j\pi/6}, \qquad a_4 = a_{-4}^* = e^{j\pi/3}, \qquad \omega_0 = \frac{2\pi}{5}$$

$$\begin{aligned}
x[n] &= 2 + a_2 e^{j2\omega_0 n} + a_{-2} e^{-j2\omega_0 n} + a_4 e^{j4\omega_0 n} + a_{-4} e^{-j4\omega_0 n} \\
&= 2 + 2e^{j\pi/6} e^{j2\omega_0 n} + 2e^{-j\pi/6} e^{-j2\omega_0 n} + e^{j\pi/3} e^{j4\omega_0 n} + e^{-j\pi/3} e^{-j4\omega_0 n} \\
&= 2 + 2\left[e^{j(2\omega_0 n + \pi/6)} + e^{-j(2\omega_0 n + \pi/6)}\right] + \left[e^{j(4\omega_0 n + \pi/3)} + e^{-j(4\omega_0 n + \pi/3)}\right] \\
&= 2 + 4\cos(2\omega_0 n + \pi/6) + 2\cos(4\omega_0 n + \pi/3) \\
&= 2 + 4\sin(2\omega_0 n + \pi/6 + \pi/2) + 2\sin(4\omega_0 n + \pi/3 + \pi/2) \\
&= 2 + 4\sin(2\omega_0 n + 2\pi/3) + 2\sin(4\omega_0 n + 5\pi/6)
\end{aligned}$$

5. (OW 3.33)

Solution

$$\frac{\partial}{\partial t} y(t) + 4y(t) = x(t)$$

assume input of form $e^{j\omega t}$ and find frequency response

$$e^{j\omega t} \longrightarrow H(j\omega)e^{j\omega t} \text{ for LTI system}$$

$$\Rightarrow \frac{\partial}{\partial t} y(t) = j\omega H(j\omega) e^{j\omega t}$$

$$\Rightarrow j\omega H(j\omega) e^{j\omega t} + 4H(j\omega) e^{j\omega t} = e^{j\omega t}$$

$$\Rightarrow H(j\omega)\left[j\omega + 4\right] = 1$$

$$H(j\omega) = \frac{1}{4 + j\omega} \quad \longleftarrow \quad \text{use this for } y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

(a) $\omega_0 = 2\pi, T = 1$

$$x(t) = \cos(2\pi t) = \frac{1}{2}\left[e^{j2\pi t} + e^{-j2\pi t}\right]$$

$$\begin{aligned}
y(t) &= \frac{1}{2}\left[H(j2\pi)e^{j2\pi t} + H(-j2\pi)e^{-j2\pi t}\right] \\
&= \frac{1}{2}\left[\frac{1}{4 + j2\pi} e^{j2\pi t} + \frac{1}{4 - j2\pi} e^{-j2\pi t}\right]
\end{aligned}$$

$$\Rightarrow b_1 = \frac{1}{2}\left(\frac{1}{4 + j2\pi}\right), \quad b_{-1} = \frac{1}{2}\left(\frac{1}{4 - j2\pi}\right)$$

(b)

$$
\begin{aligned}
x(t) &= \sin(\underbrace{4\pi t}_{T_1=1/2}) + \cos(\underbrace{6\pi t}_{T_2=1/3} + \pi/4) \\
&\Rightarrow T = 1, \omega_0 = 2\pi \\
&= \frac{1}{2j}\left[e^{j2\omega_0 t} - e^{-j2\omega_0 t}\right] + \frac{1}{2}e^{j\pi/4}e^{j3\omega_0 t} + \frac{1}{2}e^{-j\pi/4}e^{-j3\omega_0 t}
\end{aligned}
$$

$$
y(t) = \frac{1}{2j}\left[H(j4\pi)e^{j2\omega_0 t} - H(-j4\pi)e^{-j2\omega_0 t}\right] + \frac{1}{2}e^{j\pi/4}H(j6\pi)e^{j3\omega_0 t} + \frac{1}{2}e^{-j\pi/4}H(-j6\pi)e^{-j3\omega_0 t}
$$

$$
\begin{aligned}
\Rightarrow b_2 &= \frac{1}{2j}\left(\frac{1}{4+j4\pi}\right), & b_{-2} &= -\frac{1}{2j}\left(\frac{1}{4-j4\pi}\right), \\
b_3 &= \frac{1}{2j}e^{j\pi/4}\left(\frac{1}{4+j6\pi}\right), & b_{-3} &= \frac{1}{2j}e^{-j\pi/4}\left(\frac{1}{4-j6\pi}\right)
\end{aligned}
$$

6. (OW 3.36)

Solution

Similar to 3.33

$$
y[n] - \frac{1}{4}y[n-1] = x[n]
$$

$$
y[n] = H(e^{j\omega})e^{j\omega n}, \quad y[n-1] = H(e^{j\omega})e^{j\omega(n-1)}
$$

$$
H(e^{j\omega})\left[1 - \frac{1}{4}e^{-j\omega}\right]e^{j\omega n} = e^{j\omega n}
$$

$$
H(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}
$$

(a)

$$
\begin{aligned}
x[n] &= \sin\left(\frac{3\pi}{4}n\right), \quad N_0 = 8 \Rightarrow \omega_0 = \frac{2\pi}{8} = \frac{\pi}{4} \\
&= \frac{1}{2j}\left[e^{j\frac{3\pi}{4}n} - e^{-j\frac{3\pi}{4}n}\right]
\end{aligned}
$$

$$
\begin{aligned}
y[n] &= \frac{1}{2j}\left[H\left(e^{j\frac{3\pi}{4}}\right)e^{j\frac{3\pi}{4}n} - H\left(e^{-j\frac{3\pi}{4}}\right)e^{-j\frac{3\pi}{4}n}\right] \\
&= \frac{1}{2j}\left[\frac{1}{1-\frac{1}{4}e^{-j\frac{3\pi}{4}}}e^{j\frac{3\pi}{4}n} - \frac{1}{1-\frac{1}{4}e^{j\frac{3\pi}{4}}}e^{-j\frac{3\pi}{4}n}\right]
\end{aligned}
$$

$$
\Rightarrow b_3 = \frac{1}{2j}\left(\frac{1}{1-\frac{1}{4}e^{-j\frac{3\pi}{4}}}\right), \quad b_{-3} = \left(\frac{1}{2j}\frac{1}{1-\frac{1}{4}e^{j\frac{3\pi}{4}}}\right)
$$

(b)

$$x[n] = \underbrace{\cos\left(\frac{\pi}{4}n\right)}_{N_1=8} + \underbrace{2\cos\left(\frac{\pi}{2}n\right)}_{N_2=4}.$$

Therefore the period of $x[n]$ is $N = 8 \Rightarrow w_0 = \frac{2\pi}{8} = \frac{\pi}{4}$.
Following the same procedure as above

$$x[n] = \frac{1}{2}[e^{j\pi/4n} + e^{-j\pi/4n}] + [e^{j2\pi/4n} + e^{-j2\pi/4n}]$$
$$y[n] = \frac{1}{2}[\underbrace{H(e^{j\pi/4})}_{k=1}e^{j\pi/4n} + \underbrace{H(e^{-j\pi/4})}_{k=-1}e^{-j\pi/4n}] + [\underbrace{H(e^{j2\pi/4})}_{k=2}e^{j2\pi/4n} + \underbrace{H(e^{j2\pi/4})}_{k=-2}e^{-j2\pi/4n}]$$

$$\Rightarrow b_1 = \frac{1}{2}\left(\frac{1}{1-\frac{1}{4}e^{-j\pi/4}}\right), \qquad b_2 = \left(\frac{1}{1-\frac{1}{4}e^{-j\pi/2}}\right),$$
$$b_{-1} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{4}e^{j\pi/4}}\right), \qquad b_{-2} = \left(\frac{1}{1-\frac{1}{4}e^{j\pi/2}}\right)$$