

Homework #3
Due Mo. 2/25

Note: The **Basic Problems with Answers** will be worth half as much as the other questions. You must show all your work to receive credit.

1. (OW 2.8) **Basic Problem with Answer**

Solution

$$\begin{aligned}
 y(t) &= x(t) * h(t) = x(t) * [\delta(t+2) + 2\delta(t+1)] \\
 &= x(t+2) + 2x(t+1) \\
 x(t+1) &= \begin{cases} t+2 & 0 \leq t+1 \leq 1 \\ 2-(t+1) & 1 < t+1 \leq 2 \\ 0 & \text{else} \end{cases} = \begin{cases} t+2 & -1 \leq t \leq 0 \\ 1-t & 0 < t \leq 1 \\ 0 & \text{else} \end{cases} \\
 x(t+2) &= \begin{cases} t+3 & -2 \leq t \leq -1 \\ -t & -1 < t \leq 0 \\ 0 & \text{else} \end{cases} \\
 y(t) &= \begin{cases} t+3 & -2 \leq t \leq -1 \\ 2(t+2)-t & -1 < t \leq 0 \\ 2-2t & 0 < t \leq 1 \\ 0 & \text{else} \end{cases} = \begin{cases} t+3 & -2 \leq t \leq -1 \\ t+4 & -1 < t \leq 0 \\ 2-2t & 0 < t \leq 1 \\ 0 & \text{else} \end{cases}
 \end{aligned}$$

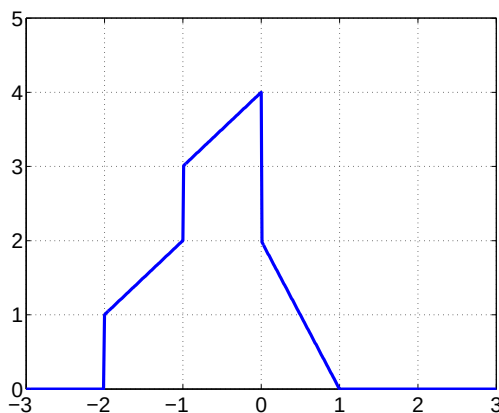


Figure 1: OW 2.8

2. (OW 2.9) **Basic Problem with Answer**

Solution

$$\begin{aligned}
 h(t) &= e^{2t}u(-t+4) + e^{-2t}u(t-5) \\
 h(t-\tau) &= e^{2(t-\tau)}u(-(t-\tau)+4) + e^{-2(t-\tau)}u(t-\tau-5) \\
 &= e^{2(t-\tau)} \underbrace{u(\tau-t+4)}_{=0 \text{ for } \tau < t-4} + e^{-2(t-\tau)} \underbrace{u(-\tau+t-5)}_{=0 \text{ for } \tau > t-5} \\
 &\Rightarrow A = t-5 \quad B = t-4
 \end{aligned}$$

3. (OW 2.10) **Basic Problem with Answer****Solution**

(a)

$$\begin{array}{ll}
 t < 0 & y(t) = 0 \\
 0 \leq t < \alpha & y(t) = \int_0^t d\tau = t \\
 \alpha \leq t < 1 & y(t) = \int_{t-\alpha}^t d\tau = t - (t - \alpha) = \alpha \\
 1 \leq t \leq 1 + \alpha & y(t) = \int_{t-\alpha}^1 d\tau = 1 - (t - \alpha) = 1 + \alpha - t \\
 t > 1 + \alpha & y(t) = 0
 \end{array}$$

$$y(t) = \begin{cases} t & 0 \leq t < \alpha \\ \alpha & 1 \leq t < 1 \\ -t + (1 + \alpha) & 1 \leq t \leq 1 + \alpha \\ 0 & \text{else} \end{cases}$$

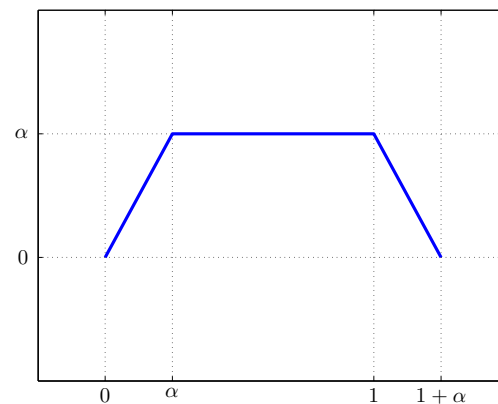


Figure 2: OW 2.10

(b) $y(t)$ has 4 discontinuities at $t = 0, \alpha, 1$, and $1 + \alpha$ unless it is a triangle $\Rightarrow \alpha = 1$.4. (OW 2.11) **Basic Problem with Answer****Solution**

(a)

$$\begin{array}{ll}
 t < 3 & y(t) = 0 \\
 3 \leq t < 5 & y(t) = \int_0^{t-3} e^{-3\tau} d\tau = -\frac{1}{3} [e^{-3\tau}]_0^{t-3} = \frac{1}{3} [1 - e^{-3(t-3)}] \\
 t \geq 5 & y(t) = \int_{t-5}^{t-3} e^{-3\tau} d\tau = -\frac{1}{3} [e^{-3\tau}]_{t-5}^{t-3} = \frac{1}{3} [e^{-3(t-5)} - e^{-3(t-3)}]
 \end{array}$$

$$y(t) = \begin{cases} \frac{1}{3} [1 - e^{-3(t-3)}] & 3 \leq t < 5 \\ \frac{1}{3} [e^{-3(t-5)} - e^{-3(t-3)}] & t \geq 5 \\ 0 & \text{else} \end{cases}$$

(b)

$$\begin{aligned} d(t) &= \frac{dx(t)}{dt} = \frac{d}{dt} [u(t-3) - u(t-5)] = \delta(t-3) - \delta(t-5) \\ g(t) &= d(t) * h(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5) \end{aligned}$$

(c)

$$\begin{aligned} \frac{d}{dt}y(t) &= \begin{cases} e^{-3(t-3)} & 3 \leq t < 5 \\ e^{-3(t-5)} - e^{-3(t-3)} & t \geq 5 \\ 0 & \text{else} \end{cases} \\ &= g(t) \end{aligned}$$

5. (OW 2.22)

Solution

(a)

$$\begin{aligned} y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau \\ &= e^{-\beta t} \int_0^t e^{(-\alpha+\beta)\tau}d\tau \quad t \geq 0 \end{aligned}$$

for $\beta = \alpha$

$$\begin{aligned} y(t) &= e^{-\beta t} \int_0^t e^0 d\tau = e^{-\beta t} [\tau]_0^t = te^{-\beta t} \\ &= te^{-\beta t}u(t) \end{aligned}$$

for $\beta \neq \alpha$

$$\begin{aligned} y(t) &= e^{-\beta t} \int_0^t e^{(-\alpha+\beta)\tau}d\tau = e^{-\beta t} \frac{1}{-\alpha+\beta} [e^{(-\alpha+\beta)\tau}]_0^t = \frac{e^{-\beta t}}{-\alpha+\beta} [e^{(-\alpha+\beta)t} - 1] \\ &= \frac{e^{-\beta t}}{-\alpha+\beta} [e^{(-\alpha+\beta)t} - 1]u(t) \end{aligned}$$

(b)

$$\begin{aligned} t > 6 \quad y(t) &= 0 \\ 3 \leq t \leq 6 \quad y(t) &= \int_{t-1}^5 (-1)e^{2(t-\tau)}d\tau = \frac{1}{2} [e^{2(t-\tau)}]_{t-1}^5 = \frac{1}{2} [e^{2(t-5)} - e^2] \\ 1 \leq t \leq 3 \quad y(t) &= \int_{t-1}^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau = -\frac{1}{2} [e^{2(t-\tau)}]_{t-1}^2 + \frac{1}{2} [e^{2(t-\tau)}]_2^5 \\ &= \frac{1}{2} [e^2 - 2e^{2(t-2)} + e^{2(t-5)}] \\ t < 1 \quad y(t) &= \int_0^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau = \frac{1}{2} [e^{2t} - 2e^{2(t-2)} + e^{2(t-5)}] \end{aligned}$$

$$y(t) = \begin{cases} \frac{1}{2} [e^{2t} - 2e^{2(t-2)} + e^{2(t-5)}] & t < 1 \\ \frac{1}{2} [e^2 - 2e^{2(t-2)} + e^{2(t-5)}] & 1 \leq t \leq 3 \\ \frac{1}{2} [e^{2(t-5)} - e^2] & 3 \leq t \leq 6 \\ 0 & \text{else} \end{cases}$$

(c)

$$\begin{aligned} t < 1 & \quad y(t) = 0 \\ 1 \leq t \leq 3 & \quad y(t) = \int_0^{t-1} (2) \sin \pi \tau d\tau = -\frac{2}{\pi} [\cos \pi \tau]_0^{t-1} = \frac{2}{\pi} [1 - \cos \pi(t-1)] \\ 3 \leq t \leq 5 & \quad y(t) = \int_{t-3}^2 (2) \sin \pi \tau d\tau = \frac{2}{\pi} [\cos \pi(t-3) - \cos \pi 2] = \frac{2}{\pi} [\cos \pi(t-3) - 1] \\ t > 6 & \quad y(t) = 0 \end{aligned}$$

$$y(t) = \begin{cases} \frac{2}{\pi} [1 - \cos \pi(t-1)] & 1 \leq t \leq 3 \\ \frac{2}{\pi} [\cos \pi(t-3) - 1] & 3 \leq t \leq 5 \\ 0 & \text{else} \end{cases}$$

(d)

Let

$$h(t) = h_1(t) - \frac{1}{3}\delta(t-2)$$

with

$$h_t(t) = \begin{cases} 4/3 & 0 \leq t \leq 1 \\ 0 & \text{else.} \end{cases}$$

$$y(t) = x(t) * h(t) = x(t) * [h_1(t) - \frac{1}{3}\delta(t-2)] = \underbrace{x(t) * h_1(t)}_{g(t)} - \frac{1}{3}x(t-2)$$

$$\begin{aligned} g(t) &= \int_{t-1}^t \left(\frac{4}{3}\right) a\tau + b d\tau = \frac{4}{3} [(1/2)a\tau^2 + b\tau]_{t-1}^t \\ &= \frac{4}{3} [(1/2)at^2 + bt - (1/2)a(t^2 - 2t + 1) - b(t-1)] \\ &= \frac{4}{3} [at + b - \frac{1}{2}a] \\ y(t) &= g(t) - \frac{1}{3}x(t-2) = \frac{4}{3} [at + b - \frac{1}{2}a] - \frac{1}{3}x(t-2) \\ &= \frac{4}{3} [at + b - \frac{1}{2}a] - \frac{1}{3} [a(t-2) + b] = at + b \\ &= x(t) \end{aligned}$$

(e) Since $x(t)$ is periodic, $y(t)$ is periodic with $T = 2$ and the convolution only needs to be computed over a single period.

$$\begin{aligned} -\frac{1}{2} \leq t \leq \frac{1}{2} & \quad y(t) = \int_{t-1}^{-1/2} (t-\tau-1)d\tau + \int_{-(1/2)}^t (1-t+\tau)d\tau = \frac{1}{4} + t - t^2 \\ \frac{1}{2} \leq t \leq \frac{3}{2} & \quad y(t) = \int_{t-1}^{1/2} (1-t+\tau)d\tau + \int_{1/2}^t (t-1-\tau)d\tau = t^2 - 3t + \frac{7}{4} \end{aligned}$$

$$y(t) = \begin{cases} \frac{1}{4} + t - t^2 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ t^2 - 3t + \frac{7}{4} & \frac{1}{2} \leq t \leq \frac{3}{2} \end{cases}$$

6. (OW 2.23)

Solution

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$y(t) = x(t) * h(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) * h(t)$$

$$= \sum_{k=-\infty}^{\infty} h(t - kT)$$

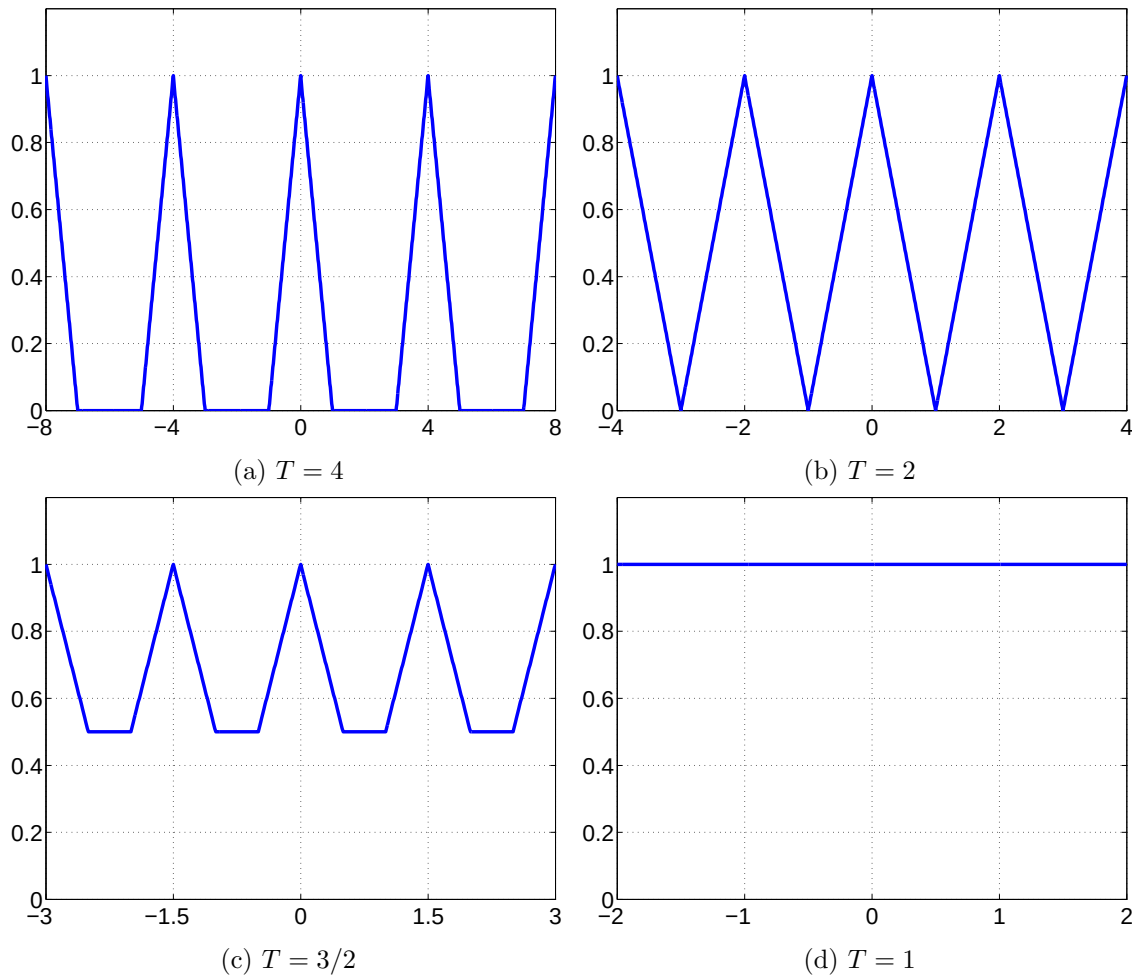


Figure 3: OW 2.23

7. (OW 2.29)

Solution

(a) Causal and stable

$$h(t) = 0 \quad \forall t < 2 \Rightarrow \text{casual}$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_2^{\infty} e^{-4t} dt = -\frac{1}{4} [e^{-4t}]_2^{\infty} = \frac{1}{4} e^{-8} < \infty \Rightarrow \text{stable}$$

(b) Not causal and not stable

$$h(t) = 0 \quad \forall t > 3 \Rightarrow \text{not casual}$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^3 e^{-6t} dt = -\frac{1}{6} [e^{-6t}]_{-\infty}^3 = \frac{1}{6} [\infty - e^{-18}] \Rightarrow \text{not stable}$$

(c) Not causal and stable

$$h(t) = 0 \quad \forall t < -50 \Rightarrow \text{not casual}$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-50}^{\infty} e^{-2t} dt = -\frac{1}{2} [e^{-2t}]_{-50}^{\infty} = \frac{1}{2} e^{100} < \infty \Rightarrow \text{stable}$$

(d) Not causal and stable

$$h(t) = 0 \quad \forall t > -1 \Rightarrow \text{not casual}$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{-1} e^{2t} dt = \frac{1}{2} [e^{2t}]_{-\infty}^{-1} = \frac{1}{2} [e^{-2} - 0] < \infty \Rightarrow \text{stable}$$

(e) Not causal and stable

$$h(t) \text{ defined } \forall t \Rightarrow \text{not casual}$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^0 e^{6t} dt + \int_0^{\infty} e^{-6t} dt = 2 \int_0^{\infty} e^{-6t} dt = 2 \frac{1}{6} = \frac{1}{3} < \infty \Rightarrow \text{stable}$$

(f) Causal and stable

$$h(t) = 0 \quad \forall t < 0 \Rightarrow \text{casual}$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_0^{\infty} t e^{-t} dt = \left[\frac{e^{-t}}{1^2} (-t - 1) \right]_0^{\infty} = [0 - (1(-1))] = 1 < \infty \Rightarrow \text{stable}$$

(g) Causal and not stable

$$h(t) = 0 \quad \forall t < 0 \Rightarrow \text{casual}$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_0^{\infty} 2e^{-t} dt + \int_0^{\infty} e^{(t-100)/100} dt = -2 [e^{-t}]_0^{\infty} + 100e^{-100} [e^{t/100}]_0^{\infty}$$

$$= 2 + 100e^{-100} [\infty - 1] \Rightarrow \text{not stable}$$

8. (OW 2.40)

Solution

(a)

$$\begin{aligned}
 y(t) &= \int_{-\infty}^t e^{-(t-\tau)} x(\tau-2) d\tau \\
 &\quad \text{let } t' = \tau - 2 \Rightarrow \tau = t' + 2 \\
 &= \int_{-\infty}^{t-2} e^{-(t-t'-2)} x(t') dt' \\
 \Rightarrow h(t) &= e^{(-t-2)} u(t-2)
 \end{aligned}$$

(b)

$$\begin{aligned}
 t < 1 & \quad y(t) = 0 \\
 1 \leq t < 4 & \quad y(t) = \int_2^{t+1} e^{-(\tau-2)} d\tau = -1 \left[e^{-(\tau-2)} \right]_2^{t+1} = 1 - e^{-(t-1)} \\
 t \geq 4 & \quad y(t) = \int_{t-2}^{t+1} e^{-(\tau-2)} d\tau = -1 \left[e^{-(\tau-2)} \right]_{t-2}^{t+1} = e^{-(t-4)} - e^{-(t-1)}
 \end{aligned}$$

$$y(t) = \begin{cases} 0 & t < 1 \\ 1 - e^{-(t-1)} & 1 \leq t < 4 \\ e^{-(t-4)} - e^{-(t-1)} & t \geq 4 \end{cases}$$

9. (OW 2.46)

Solution

$$\begin{aligned}
 x(t) &= 2e^{-3t} u(t-1) \longrightarrow y(t) \\
 \frac{dx(t)}{dt} &\longrightarrow -3y(t) + e^{-2t} u(t)
 \end{aligned}$$

$$\begin{aligned}
 \frac{dx(t)}{dt} &= 2e^{-3t} \delta(t-1) + -6e^{-3t} u(t-1) \\
 &= 2e^{-3} \delta(t-1) - 6e^{-3t} u(t-1) \\
 &= 2e^{-3} \delta(t-1) - 3x(t)
 \end{aligned}$$

$$\Rightarrow -3x(t) + 2e^{-3} \delta(t-1) \longrightarrow -3y(t) + e^{-2t} u(t)$$

since $-3x(t) \longrightarrow -3y(t)$ by definition and linearity

$$\begin{aligned}
 \underbrace{2e^{-3}}_a \delta(t-1) &\longrightarrow e^{-2t} u(t) \\
 a\delta(t-1) &\longrightarrow e^{-2t} u(t) = ah(t-1)
 \end{aligned}$$

$$\begin{aligned}\Rightarrow h(t-1) &= \frac{1}{a} e^{-2t} u(t) = \frac{1}{2e^{-3}} e^{-2t} u(t) \\ &= \frac{1}{2} e^{-2t+3} u(t)\end{aligned}$$

$$\text{let } \tau = t - 1$$

$$\begin{aligned}h(\tau) &= \frac{1}{2} e^{-2(\tau+1)+3} u(\tau+1) \\ &= \frac{1}{2} e^{-2\tau+1} u(\tau+1)\end{aligned}$$