Homework #2
Due Mo. 2/11

Note: The **Basic Problems with Answers** will be worth half as much as the other questions. You must show all your work to receive credit.

1. (OW 1.27 (a),(b),(e))

   (a) Linear and stable

   (1) [Not memoryless] For $t = 0$ need $x(-2)$

   (2) [Not TI]

$$\begin{aligned} y_1(t) &= y(t - t_0) = x(t - t_0 - 2) + x(2 - (t - t_0)) \\ &= x(t - t_0 - 2) + x(2 - t + t_0)) \\ y_2(t) &= f(x(t - t_0)) = x(t - t_0 - 2) + x(2 - t - t_0) \\ y_1(t) &\neq y_2(t) \Rightarrow \text{not TI} \end{aligned}$$

   (3) [Linear]

$$\begin{aligned} x_3(t) &= ax_1(t) + bx_2(t) \\ y_3(t) &= x_3(t - 2) + x_3(2 - t) \\ &= ax_1(t - 2) + bx_2(t - 2) + ax_1(2 - t) + bx_2(2 - t) \\ &= a[x_1(t - 2) + x_1(2 - t)] + b[x_2(t - 2) + x_2(2 - 2)] \\ &= ay_1(t) + by_2(t) \Rightarrow \text{linear} \end{aligned}$$

   (4) [Not causal] For $t = 0$ need $x(-2)$ and $x(2)$

   (5) [Stable] Assume $|x(t)| < \beta$

$$\begin{aligned} |y(t)| &= |x(t - 2) + x(2 - t)| \\ &\leq |x(t - 2)| + |x(2 - t)| \\ &\leq 2\beta \Rightarrow \text{stable} \end{aligned}$$

   (b) Memoryless, linear, causal, and stable

   (1) [Memoryless] $\cos(3t)$ is a constant

   (2) [Not TI]

$$\begin{aligned} y_1(t) &= y(t - t_0) = [\cos(3(t - t_0)]x(t - t_0) = \cos(3t - 3t_0)x(t - t_0) \\ y_2(t) &= f(x(t - t_0)) = \cos(3t)x(t - t_0) \\ y_1(t) &\neq y_2(t) \Rightarrow \text{not TI} \end{aligned}$$

   (3) [Linear]

$$\begin{aligned} x_3(t) &= ax_1(t) + bx_2(t) \\ y_3(t) &= \cos(3t)x_3(t) \\ &= \cos(3t)[ax_1(t) + bx_2(t)] \\ &= ay_1(t) + by_2(t) \Rightarrow \text{linear} \end{aligned}$$

(4) [Causal] Since $y(t)$ only depends on time $t$, memoryless $\Rightarrow$ causal

(5) [Stable] Assume $|x(t)| < \beta$

$$\begin{aligned}|y(t)| &= |\cos(3t)x(t)| \\ &\leq |\cos(3t)||x(t)| \\ &\leq |x(t)| \\ &\leq \beta \Rightarrow \text{stable}\end{aligned}$$

(e) Time-invariant, linear, causal, and stable

(1) [Not memoryless] Depends on $x(t-2)$ term.

(2) [TI]

$$\begin{aligned}y_1(t) &= y(t-t_0) = \begin{cases} 0 & x(t-t_0) < 0 \\ x(t-t_0) + x(t-t_0-2) & x(t-t_0) \geq 0 \end{cases} \\ y_2(t) &= f(x(t-t_0)) = \begin{cases} 0 & x(t-t_0) < 0 \\ x(t-t_0) + x(t-t_0-2) & x(t-t_0) \geq 0 \end{cases} \\ y_1(t) &= y_2(t) \Rightarrow \text{TI}\end{aligned}$$

(3) [Linear]

$$\begin{aligned}x_3(t) &= ax_1(t) + bx_2(t) \\ y_3(t) &= \begin{cases} 0 & x_3(t) < 0 \\ x_3(t) + x_3(t-2) & x_3(t) \geq 0 \end{cases} \\ &= \begin{cases} 0 & x_3(t) < 0 \\ ax_1(t) + bx_2(t) + ax_1(t-2) + bx_2(t-2) & x_3(t) \geq 0 \end{cases} \\ &= \begin{cases} 0 & x_3(t) < 0 \\ ax_1(t) + ax_1(t-2) + bx_2(t) + bx_2(t-2) & x_3(t) \geq 0 \end{cases} \\ &= \begin{cases} 0 & x_3(t) < 0 \\ ay_1(t) + by_2(t) & x_3(t) \geq 0 \end{cases} \\ &\Rightarrow \text{linear}\end{aligned}$$

(4) [Causal] Since $\forall t_0, y(t_0)$ only depends on time $t \leq t_0$

(5) [Stable] Assume $|x(t)| < \beta$

$$\begin{aligned}|y(t)| &= |x(t) + x(t-2)| \\ &\leq |x(t)| + |x(t-2)| \\ &\leq 2\beta \Rightarrow \text{stable}\end{aligned}$$

2. (OW 1.28 (a),(b),(f))

**Solution**

(a) Linear and stable

(1) [Not memoryless] For $n = 2$ need $x[-2]$

(2) [Not TI]

$$y_1[n] = y[n-n_0] = x[-(n-n_0)] = x[-n+n_0]$$
$$y_2[n] = f(x[n-n_0]) = x[-n-n_0]$$
$$y_1[n] \neq y_2[n] \Rightarrow \text{not TI}$$

(3) [Linear]

$$\begin{aligned} y_3[n] &= x_3[-n] \\ &= ax_1[-n] + bx_2[-n] \\ &= ay_1[n] + by_2[n] \Rightarrow \text{linear} \end{aligned}$$

(4) [Not causal] For $n = -2$ need $x[2]$

(5) [Stable] Assume $|x[n]| < \beta$

$$|y[n]| = |x[-n]| \leq \beta \Rightarrow \text{stable}$$

(b) TI, linear, causal, stable

(1) [Not memoryless] Has $x[n-2]$ term

(2) [TI]

$$y_1[n] = y[n-n_0] = x[n-n_0-2] - x[n-n_0-8]$$
$$y_2[n] = f(x[n-n_0]) = x[n-n_0-2] - x[n-n_0-8]$$
$$y_1[n] = y_2[n] \Rightarrow \text{TI}$$

(3) [Linear]

$$\begin{aligned} y_3[n] &= ax_1[n-2] + bx_2[n-2] - 2(ax_1[n-8] + bx_2[n-8]) \\ &= a(x_1[n-2] - 2x_1[n-8]) + b(x_2[n-2] - 2x_2[n-8]) \quad = ay_1[n] + by_2[n] \Rightarrow \text{linear} \end{aligned}$$

(4) [Causal] $\forall t_0, y[t_0]$ only depends on $t < t_0$

(5) [Stable] Assume $|x[n]| < \beta$

$$\begin{aligned} |y[n]| &= |x[n-2] - 2x_1[n-8]| \\ &\leq |x[n-2]| + 2|x_1[n-8]| \leq 3\beta \Rightarrow \text{stable} \end{aligned}$$

(f) Memoryless, linear, causal, stable

(1) [Memoryless] Only depends on $x[n]$.

(2) [Not TI]

$$y_1[n] = y[n-n_0] = \begin{cases} x[n-n+0] & n-n_0 \geq 1 \\ 0 & n-n_0 = 0 \\ x[n-n+0] & n-n_0 \neq -1 \end{cases}$$

$$y_2[n] = f(x[n-n_0]) = \begin{cases} x[n-n+0] & n \geq 1 \\ 0 & n = 0 \\ x[n-n+0] & n \neq -1 \end{cases}$$

$$y_1[n] \neq y_2[n] \Rightarrow \text{not TI}$$

(3) [Linear]

$$x_3[n] = ax_1[n] + bx_2[n]$$

$$\begin{aligned} y_3[n] &= \begin{cases} x_3[n] & n \geq 1 \\ 0 & n = 0 \\ x_3[n] & n \neq -1 \end{cases} \\ &= \begin{cases} ax_1[n] + bx_2[n] & n \geq 1 \\ 0 & n = 0 \\ ax_1[n] + bx_2[n] & n \neq -1 \end{cases} \\ &= y_1[n] + y_2[n] \Rightarrow \text{linear} \end{aligned}$$

(4) [Causal] Memoryless $\Rightarrow$ causal

(5) [Stable] Assume $|x(t)| < \beta$

$$\begin{aligned} |y[n]| &= \begin{cases} |x[n]| & n \geq 1 \\ 0 & n = 0 \\ |x[n]| & n \neq -1 \end{cases} \\ &< \beta \forall n \Rightarrow \text{stable} \end{aligned}$$

3. (OW 2.2) **Basic Problem with Answer**

**Solution**

$$\begin{aligned} h[n] &= \left(\frac{1}{2}\right)^{n-1} u[n+3] - u[n-10] \\ &= \begin{cases} (1/2)^{n-1} & -3 \leq n \leq 9 \\ 0 & \text{else} \end{cases} \\ h[n-k] &= \begin{cases} (1/2)^{n-k-1} & -3 \leq n-k \leq 9 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} (1/2)^{n-k-1} & n-9 \leq k \leq n+3 \\ 0 & \text{else} \end{cases} \end{aligned}$$

Therefore

$$A = n - 9 \qquad B = n + 3$$

4. (OW 2.4) **Basic Problem with Answer**

**Solution**

$$x[n] = \begin{cases} 1 & 3 \leq n \leq 8 \\ 0 & \text{else} \end{cases} \qquad h[n] = \begin{cases} 1 & 4 \leq n \leq 15 \\ 0 & \text{else} \end{cases}$$

$$
\begin{aligned}
n < 7 \qquad & y[n] = 0 \\
7 \leq n \leq 12 \qquad & y[n] = \sum_{4}^{n-3} 1 = n - 3 - 4 + 1 = n - 6 \\
12 \leq n \leq 18 \qquad & y[n] = \sum_{n-8}^{n-3} 1 = n - 3 - (n - 8) + 1 = 6 \\
18 \leq n \leq 23 \qquad & y[n] = \sum_{n-8}^{15} 1 = 15 - (n - 8) + 1 = 24 - n \\
n > 23 \qquad & y[n] = 0
\end{aligned}
$$

$$
y[n] = \begin{cases} 0 & n < 7 \\ n - 6 & 7 \leq n \leq 12 \\ 6 & 12 \leq n \leq 18 \\ 24 - n & 18 \leq n \leq 23 \\ 0 & n > 23 \end{cases}
$$

5. (OW 2.21)

   **Solution**

   (a)

$$
\begin{aligned}
y[n] &= x[n] * h[n] = \alpha^n u[n] * \beta^n u[n] \\
&= \sum_{k=0}^{n} \alpha^k \beta^{n-k} = \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \\
&= \beta^n \left(\frac{1 - (\alpha/\beta)^{n+1}}{1 - (\alpha/\beta)}\right) u[n]
\end{aligned}
$$

   (b)

$$
\begin{aligned}
y[n] &= x[n] * h[n] = \alpha^n u[n] * \alpha^n u[n] \\
&= \sum_{k=0}^{n} \alpha^k \alpha^{n-k} = \alpha^n \sum_{k=0}^{n} 1 \\
&= \alpha^n (n + 1) u[n]
\end{aligned}
$$

   (c)

$$
y[n] = x[n] * h[n] = (-1/2)^n u[n - 4] * 4^n u[2 - n]
$$

$$n \le 6 \qquad y[n] = \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)^k (4)^{n-k} = 4^n \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^k$$

$$= 4^n \sum_{k=4}^{\infty} \left(-\frac{1}{8}\right)^k = 4^n \left(-\frac{1}{8}\right)^4 \left(\frac{1}{1-\frac{1}{8}}\right)$$

$$= \left(\frac{8}{9}\right) \left(-\frac{1}{8}\right)^4 4^n$$

$$n > 6 \qquad y[n] = \sum_{k=n-2}^{\infty} \left(-\frac{1}{2}\right)^k (4)^{n-k}$$

$$= 4^n \sum_{k=n-2}^{\infty} \left(-\frac{1}{8}\right)^k$$

$$= \left(\frac{8}{9}\right) \left(-\frac{1}{8}\right)^{n-2} 4^n$$

(d) See Figure 1.

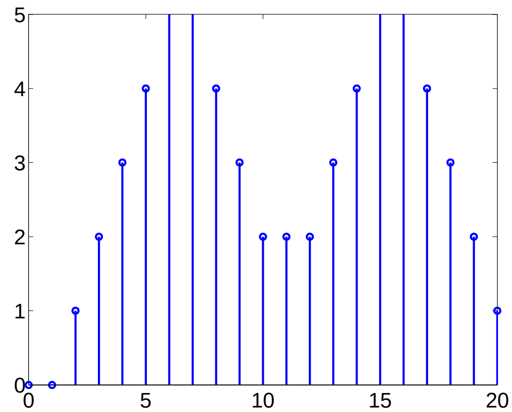


Figure 1: OW 2.21 (d) Solution

6. (OW 2.24)

**Solution**

(a)

$$
\begin{aligned}
h_2[n] &= u[n] - u[n-2] = \delta[n] + \delta[n-1] \\
h_3[n] &= h_2[n] * h_2[n] \\
&= (\delta[n] + \delta[n-1]) * (\delta[n] + \delta[n-1]) \\
&= \delta[n] + 2\delta[n-1] + \delta[n-2] \\
h[n] &= h_1[n] * h_3[n] \\
&= h_1[n] + 2h_1[n-1] + h_1[n-2]
\end{aligned}
$$

$h_1[n]$ can be found recursively:

| | | | | |
|---|---|---|---|---|
| $n = 0$ | $h[0] = h_1[0]$ | $= 1$ | $\Rightarrow$ | $h_1[0] = 1$ |
| $n = 1$ | $h[1] = h_1[1] + 2h_1[0]$ | $= 5$ | $\Rightarrow$ | $h_1[1] = 3$ |
| $n = 2$ | $h[2] = h_1[2] + 2h_1[1] + h_1[0]$ | $= 10$ | $\Rightarrow$ | $h_1[2] = 3$ |
| $n = 3$ | $h[3] = h_1[3] + 2h_1[2] + h_1[1]$ | $= 11$ | $\Rightarrow$ | $h_1[3] = 2$ |
| $n = 4$ | $h[4] = h_1[4] + 2h_1[3] + h_1[2]$ | $= 11$ | $\Rightarrow$ | $h_1[4] = 1$ |
| $n = 5$ | $h[5] = h_1[5] + 2h_1[4] + h_1[3]$ | $= 11$ | $\Rightarrow$ | $h_1[5] = 0$ |

(b)

$$x[n] = \delta[n] - \delta[n-1] \longrightarrow y[n] = h[n] - h[n-1]$$

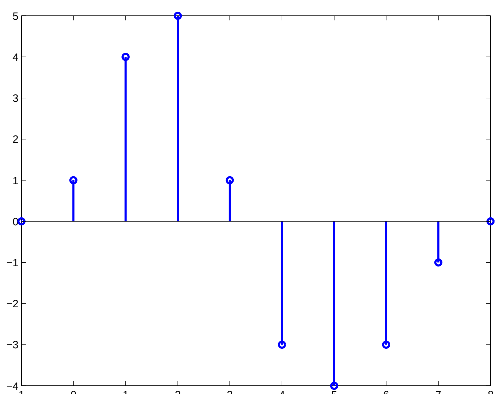


Figure 2: OW 2.24 (b) Solution

7. (OW 2.26)

**Solution**

(a)

$$\begin{aligned} g[n] &= x_1[n] * x_2[n] = (0.5)^n u[n] * u[n+3] \\ &= \begin{cases} \sum_{k=0}^{n+3} \left(\frac{1}{2}\right)^k & n \geq -3 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{1 - 0.5^{n+4}}{1 - 0.5} & n \geq -3 \\ 0 & \text{else} \end{cases} \\ &= 2(1 - 0.5^{n+4})u[n+3]. \end{aligned}$$

(b)

$$y[n] = g[n] * x_3[n] = g[n] * [\delta[n] - \delta[n-1]] = g[n] - g[n-1]$$

(c)

$$x_4[n] = x_2[n] * x_3[n] = u[n+3] - u[n+2] = \delta[n+3]$$

(d)

$$x_5[n] = x_1[n] * x_4[n] = (0.5)^{n+3} u[n+3]$$

8. (OW 2.51)

**Solution**

(a) System A - System B

$$\delta[n] \xrightarrow{A} h[n] = \left(\frac{1}{2}\right)^n u[n] \xrightarrow{B} n\left(\frac{1}{2}\right)^n u[n]$$

System B - System A

$$\delta[n] \xrightarrow{B} n\delta[n] = 0 \xrightarrow{A} 0$$

(b) System A - System B

$$\delta[n] \xrightarrow{A} \left(\frac{1}{2}\right)^n u[n] \xrightarrow{B} \left(\frac{1}{2}\right)^n u[n] + 2$$

System B - System A

$$\begin{aligned} \delta[n] \xrightarrow{B} \delta[n] + 2 \xrightarrow{A} y[n] &= h[n] * (\delta[n] + 2) \\ &= h[n] + h[n] * 2 \\ &= h[n] + 2\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \\ &= h[n] + 2\frac{1}{1 - 0.5} \\ &= h[n] + 4 \end{aligned}$$