

Appendix A: Partial Fraction Expansion

1 Overview

Partial fraction expansion (PFE) is quite useful in LTI system analysis.

We know

$$\begin{aligned}y(t) &= x(t) * h(t) \\Y(s) &= X(s)H(s) \\ \Rightarrow H(s) &= \frac{Y(s)}{X(s)}.\end{aligned}$$

Thus, the transfer function is a rational polynomial function.

Steps for solving an LTI system:

1. Compute $H(s)$ or $Y(s)$ as a rational function
2. Use PFE to express $H(s)$ in a simple form
3. Inverse transform partial fraction terms using lookup table (e.g. Table 9.2)

Example

Find $y(t)$ given

$$x(t) = e^{-bt}u(t), \quad b > 0 \qquad h(t) = e^{-at}u(t), \quad a > 0.$$

Take FT and perform PFE

$$\begin{aligned}Y(j\omega) &= X(j\omega)H(j\omega) = \left(\frac{1}{b+j\omega}\right) \left(\frac{1}{a+j\omega}\right) = \frac{1}{(b+j\omega)(a+j\omega)} \\ &= \left(\frac{A}{b+j\omega}\right) + \left(\frac{B}{a+j\omega}\right) \\ &= \left(\frac{\frac{1}{a-b}}{b+j\omega}\right) + \left(\frac{\frac{-1}{a+b}}{a+j\omega}\right)\end{aligned}$$

Take inverse transform

$$y(t) = \frac{1}{a-b}e^{-bt}u(t) - \frac{1}{a+b}e^{-at}u(t)$$

Since the basic steps to solve systems appears often (FT, Laplace, Z), it is essential to know how to do PFE. Notice in this example we used the Fourier Transform since it is a little easier without looking at ROC.

2 Criteria for PFE

The denominator should be higher-order than the numerator otherwise you must do polynomial long division.

Reminder for polynomial long division: Assume you have the rational polynomial below

$$G(v) = \frac{2v^2 + v + \gamma_0}{v + 2}.$$

This requires polynomial long division before PFE since the order of the numerator is two (v^2) and the denominator is one (v).

$$\begin{array}{r} 2v - 3 \\ v + 2 \overline{) 2v^2 + v + 5} \\ \underline{- 2v^2 - 4v} \\ - 3v + 5 \\ \underline{3v + 6} \\ 11 \end{array}$$

This results in simplified equation

$$G(v) = 2v - 3 + \frac{11}{v + 2}$$

Note: when doing the long division, it should be repeated until the numerator (remainder) is of lower order than denominator. In this case, since the denominator is order one, the remainder had to be order 0 (constant). Once the numerator is of lower order, PFE can be performed on the resulting rational polynomial. This was not needed in this case since the denominator is order 1 but would have been needed if order 2.

3 Heavyside Cover-up

Need to have a rational function with numerator of lower degree than denominator and have a factored form

$$G(v) = \frac{b_{n-1}v^{n-1} + \dots + b_1v + b_0}{(v - p_1)^{\sigma_1}(v - p_2)^{\sigma_2} \dots (v - p_r)^{\sigma_r}},$$

where $r \geq n$, p_i are the roots, and σ_i is the multiplicity of a root (PFE is more tricky for $\sigma_i > 1$).

Second Order Example

$$G(v) = \frac{b_2v^2 + b_1v + b_0}{(v - p_1)^2(v - p_2)} = \underbrace{\frac{A}{(v - p_1)^2}}_{\textcircled{1}} + \underbrace{\frac{B}{(v - p_1)}}_{\textcircled{2}} + \underbrace{\frac{C}{(v - p_2)}}_{\textcircled{3}}.$$

Note: polynomial long division is not needed in this case since the numerator is order 2 and denominator is order 3. Repeated roots require a partial term for each order of the root. In this case, $(v - p_1)^2$ resulted in a partial term of $(v - p_1)^2$ and $(v - p_1)$ to handle $\sigma_i = 1$ to $\sigma_i = 2$ in this second order case.

In this form it is possible to equate the numerators and solve for A, B, C with a system of equations in the standard PFE process you learned in math

$$b_2v^2 + b_1v + b_0 = A(v - p_2) + B(v - p_1)(v - p_2) + C(v - p_1)^2.$$

Instead of the system of equations, we will use the heavyside cover-up shortcut.

① When solving this, attack the highest order σ_i first.

$$(v - p_1)^2 G(v) = A + B(v - p_1) + \frac{C(v - p_1)^2}{v - p_2}.$$

Examining the above equation you will notice that you can isolate A from B, C by letting $v = p_1$ which will result in zero multiplied by B and C .

$$(v - p_1)^2 G(v) \Big|_{v=p_1} = A + B \times 0 + \frac{C \times 0^2}{v - p_2}$$

Therefore, when solving for A ,

$$\begin{aligned} \Rightarrow A &= (v - p_1)^2 G(v) \Big|_{v=p_1} = (v - p_1)^2 \left[\frac{b_2 v^2 + b_1 v + b_0}{(v - p_1)^2 (v - p_2)} \right] \\ &= \frac{b_2 v^2 + b_1 v + b_0}{v - p_2} \Big|_{v=p_1} \end{aligned}$$

② Now look at the $(v - p_1)$ term. If you try the same technique as ① above

$$(v - p_1) G(v) = \frac{A}{v - p_1} + B + \frac{C(v - p_1)}{v - p_2}.$$

This will not isolate B with $v = p_1$ because of the denominator $(v - p_1)$ term will cause an undefined term.

However, looking at ①, notice

$$\begin{aligned} \frac{d}{dv} [(v - p_1)^2 G(v)] &= \frac{d}{dv} \left[A + B(v - p_1) + \frac{C(v - p_1)^2}{v - p_2} \right] \\ &= B + C \underbrace{\left[\frac{2(v - p_1)(v - p_2) - (v - p_1)^2}{(v - p_2)^2} \right]}_{=0 \text{ when } v=p_1} \end{aligned}$$

Therefore, we can solve for B (the ② term) as

$$\begin{aligned} \Rightarrow B &= \frac{d}{dv} [(v - p_1)^2 G(v)] = \frac{b_2 v^2 + b_1 v + b_0}{v - p_2} \Big|_{v=p_1} \\ &= \frac{(2b_2 v + b_1)(v - p_2) - (b_2 v^2 + b_1 v + b_0)(1)}{(v - p_2)^2} \Big|_{v=p_1}. \end{aligned}$$

③ Finally, look at the last $(v - p_2)$ term,

$$\begin{aligned} (v - p_2) G(v) &= \frac{(v - p_2)A}{(v - p_1)^2} + \frac{(v - p_2)B}{(v - p_1)} + C \\ \Rightarrow C &= (v - p_2) G(v) \Big|_{v=p_2} = \frac{b_2 v^2 + b_1 v + b_0}{(v - p_1)^2} \Big|_{v=p_2} \end{aligned}$$

Example 4.25

Find the impulse response $h(t)$ given the following differential system

$$\begin{aligned}\mathcal{F}\left\{\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t)\right\} &= \mathcal{F}\left\{\frac{dx(t)}{dt} + 2x(t)\right\} \\ (j\omega)^2Y(j\omega) + 4j\omega Y(j\omega) + 3Y(j\omega) &= j\omega X(j\omega) + 2X(j\omega) \\ Y(j\omega) [(j\omega)^2 + 4j\omega + 3] &= X(j\omega) [j\omega + 2]\end{aligned}$$

$$\begin{aligned}\Rightarrow H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} \\ &= \frac{j\omega + 2}{(j\omega + 3)(j\omega + 1)} = \frac{A}{j\omega + 3} + \frac{B}{j\omega + 1}\end{aligned}$$

Solve for the partial fraction terms

$$\begin{aligned}A &= (j\omega + 3)H(j\omega)|_{j\omega=-3} = \frac{(j\omega + 2)(j\omega + 3)}{(j\omega + 3)(j\omega + 1)} \Big|_{j\omega=-3} \\ &= \frac{j\omega + 2}{j\omega + 1} \Big|_{j\omega=-3} = \frac{-1}{-2} = \frac{1}{2} \\ B &= (j\omega + 1)H(j\omega)|_{j\omega=-1} = \frac{j\omega + 2}{j\omega + 3} \Big|_{j\omega=-1} = \frac{1}{2}\end{aligned}$$

This results in the full PFE of $H(j\omega)$ as

$$H(j\omega) = \frac{1/2}{j\omega + 3} + \frac{1/2}{j\omega + 1}$$

The impulse response can be found by inverse FT

$$\begin{aligned}h(t) &= \mathcal{F}^{-1}\{H(j\omega)\} = \mathcal{F}^{-1}\left\{\frac{1/2}{j\omega + 3} + \frac{1/2}{j\omega + 1}\right\} \\ &= \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)\end{aligned}$$

4 Discrete Time PFE

While most things are the same, there are some subtle differences for PFE in DT. Given a rational system function

$$G(v) = \frac{d_{n-1}v^{n-1} + \dots + d_1v + d_0}{f_nv^n + \dots + f_1v + 1}$$

Notice the numerator degree is less than denominator. This can be factored as

$$G(v) = \frac{d_{n-1}v^{n-1} + \dots + d_1v + d_0}{(1 - p_1^{-1}v)^{\sigma_1}(1 - p_2^{-1}v)^{\sigma_2} \dots (1 - p_r^{-1}v)^{\sigma_r}}$$

which can be written in PFE as

$$G(v) = \sum_{i=1}^r \sum_{k=1}^{\sigma_i} \frac{B_{ik}}{(1 - p_i^{-1}v)^k}.$$

With heavyside cover-up the solutions can be found as

$$B_{ik} = \frac{1}{(\sigma_i - k)!} (-p_i)^{\sigma_i - k} \left\{ \frac{d^{\sigma_i - k}}{dv^{\sigma_i - k}} [(1 - p_i^{-1}v)^{\sigma_i} G(v)] \right\} \Big|_{v=p_i}$$

For our most often occurring cases of single and double root the solutions are below.

Single root ($\sigma_i = 1$) \Rightarrow same as CT case

$$B_{i1} = (1 - p_i^{-1})G(v) \Big|_{v=p_i}$$

Double root ($\sigma_i = 1$) \Rightarrow subtle differences from p_i^{-1} term

$$(\text{repeated}) \quad B_{i2} = (1 - p_i^{-1})^2 G(v) \Big|_{v=p_i}$$

$$(\text{single}) \quad B_{i1} = (-p_i) \left[\frac{d}{dv} (1 - p_i^{-1})^2 G(v) \right] \Big|_{v=p_i}.$$

Notice in the repeated root section, the single power has an extra $-p_i$ term out in front.

Example

$$\begin{aligned} h[n] &= 4 \left(\frac{1}{2} \right)^n u[n] - 2 \left(\frac{1}{4} \right)^n u[n] & x[n] &= \left(\frac{1}{4} \right)^n u[n] \\ H(z) &= \frac{2}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} & X(z) &= \frac{1}{1 - \frac{1}{4}z^{-1}} \\ Y(z) &= X(z)H(z) \\ &= \frac{2}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})^2} \\ &= \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B}{(1 - \frac{1}{4}z^{-1})^2} + \frac{C}{1 - \frac{1}{2}z^{-1}} \end{aligned}$$

Using normal heavyside cover-up can find C and B .

$$\begin{aligned} C &= \left(1 - \frac{1}{2}z^{-1} \right) Y(z) \Big|_{z^{-1}=2} = \frac{2}{(1 - \frac{1}{4}z^{-1})^2} \Big|_{z^{-1}=2} = \frac{2}{(1 - \frac{1}{2})^2} = \frac{1}{1/4} = 8 \\ B &= \left(1 - \frac{1}{4}z^{-1} \right) Y(z) \Big|_{z^{-1}=4} = \frac{2}{(1 - \frac{1}{2}z^{-1})^2} \Big|_{z^{-1}=4} = \frac{2}{1 - 2} = -2 \end{aligned}$$

The first-order single root term A is slightly more complicated. First note and define the pole term for A is $p_A^{-1} = \frac{1}{4}$.

$$\begin{aligned} A &= -p_A \left[\frac{d}{dz^{-1}} (1 - p_A^{-1}z^{-1})^2 Y(z) \right] \Big|_{z^{-1}=p_A} = -4 \frac{d}{dz^{-1}} \left(1 - \frac{1}{4}z^{-1} \right) Y(z) \Big|_{z^{-1}=4} \\ &= -4 \frac{d}{dz^{-1}} 2 \left(1 - \frac{1}{2}z^{-1} \right)^2 \Big|_{z^{-1}=4} = -4 \frac{d}{dz^{-1}} 2 \left(1 - \frac{1}{2}z^{-1} \right)^{-1} \\ &= -4 \left[-2 \left(1 - \frac{1}{2}z^{-1} \right)^{-2} \left(-\frac{1}{2} \right) \right] \Big|_{z^{-1}=4} \\ &= -4 \frac{1}{(1 - \frac{1}{2}z^{-1})^2} \Big|_{z^{-1}=4} = \frac{-4}{(1 - 2)^2} = -4 \end{aligned}$$

Note that it may be more simple to not carry around the z^{-1} terms (or $e^{-j\omega}$ for FT) since it might make the derivative more difficult to keep track of. Instead do a substitution of $v = z^{-1}$ early on to solve with $\frac{d}{dv}$.