EE360: Signals and System I

Schaum’s Signals and Systems
Chapter 7
State Space Analysis

http://www.ee.unlv.edu/~b1morris/ee360/
Outline

- Concept of State
- DT State Space Representations
- DT State Equation Solutions
- CT State Space Representations
- CT State Equation Solutions
Introduction to State Space

- So far, we have studied LTI systems based on input-output relationships
  - Known as external description of a system
- Now will examine state space representation of systems
  - Known as internal description of systems

- Consists of two parts
  - State equations – set of equations relating state variables to inputs
  - Output equations – set of equations relating outputs to state variables and inputs
Advantages of State Space

- Provides new insight into system behavior
  - Use of matrix linear algebra
- Can handle multiple-input multiple-output (MIMO) systems
  - Generalize from single-input single-output
- Can be extended to non-linear and time-varying systems
  - General mathematical model
- State equations can be implemented efficiently on computers
  - Enables solving/simulation of complex systems
Definition of State

- State – state of a system @ time $t_0$ is defined as the *minimal information* that is *sufficient* to determine the *state* and *output* of a system for all times $t > t_0$ when the input is also known for $t > t_0$

- State variables ($q_i$) - variables that contain all state information (memory)

- Note: definition only applies to causal systems
Motivation Example

• Input: \( v(t) \)
• Output: \( i(t) \)
• State:
  - \( v_L(t) = L \frac{di}{dt} \)
  - \( i_c(t) = C \frac{dv_c}{dt} \)

• Knowing \( x(t) = v(t) \) over \([-\infty, t] \) is sufficient to determine \( y(t) = i(t) \) over the same interval

• If \( x(t) \) is only known between \([t_0, t] \) then the output cannot be determined without knowledge of
  - Current through inductor
  - Voltage across capacitor

• Imagine being handed a circuit (system) that was in operation at time \( t_0 \)
  - Initial condition problem
Selection of State Variables

• Need to determine “memory elements” of a system

• DT: Select outputs of delay elements
• CT: Select outputs of integrators or energy-storing elements (capacitors, inductors)

• However, state-variable choice is not unique
  ▫ Transformations of variables will result in same state space analysis
DT State Space Representation I

- Consider a single-input single-output (SISO) DT LTI system
  - $y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = x[n]$
- To uniquely determine a complete solution (output), requires $N$ initial conditions
  - $y[-1], y[-2], \ldots, y[-N]$
- Define state variables (outputs of delay elements)
  - $q_1[n] = y[n-N]$
  - $q_2[n] = y[n-(N-1)] = y[n-N+1]$
  - $\cdots$
  - $q_N[n] = y[n-1]$

\[ N \text{ state vars} \]
DT State Space Representation II

- Find next (step-ahead) state
  - By definition of delay or using signal flow graph

- \( q_1[n + 1] = y[n + 1 - N] = q_2[N] \)
- \( q_2[n + 1] = y[n + 1 - N + 1] = y[n - N + 2] = q_3[n] \)
- ...
- \( q_N[n + 1] = y[n + 1 - 1] = y[n] = -a_1 y[n - 1] + \cdots + -a_N y[n - N] \) (recursive form)

- \( q_N[n + 1] = -a_1 q_N[n] - a_2 q_{N-1}[n] + \cdots + -a_N q_1[n] \)
These relationships can be compactly expressed in matrix form

\[
\begin{bmatrix}
q_1[n+1] \\
q_2[n+1] \\
\vdots \\
q_N[n+1]
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
-a_N & -a_{N-1} & -a_{N-2} & -a_{N-3} & \cdots & -a_1
\end{bmatrix}
\begin{bmatrix}
q_1[n] \\
q_2[n] \\
\vdots \\
q_N[n]
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix}x[n]
\]

\[
y[n] = 
\begin{bmatrix}
-a_N & -a_{N-1} & \cdots & -a_1
\end{bmatrix}
\begin{bmatrix}
q_1[n] \\
q_2[n] \\
\vdots \\
q_N[n]
\end{bmatrix}
+ 
\begin{bmatrix}
1
\end{bmatrix}x[n]
\]

State equation – next state from past state and input

\[q[n + 1] = Aq[n] + Bx[n]\]

Output equation – output based on state and input

\[y[n] = Cq[n] + Dx[n]\]

Note: generalized form for MIMO systems with vector \(x[n], y[n]\)
DT State Space Representation IV

• Previous example:
  ▫ Defined state variables as outputs of delay elements
  ▫ Rewrote state relationships using a vectorized form of state $q[n]$

• Goal: build state-equations given either a difference equation or block-diagram

• Note: previous example had no delayed input $x[n]$. How would delayed inputs change state space representation?
  ▫ Consider DFII structure and develop state equations
Similarity Transformation

- Choice of state-variable is not unique

- Can have another choice of state variables as a transformation
- If $\mathbf{v}[n] = T\mathbf{q}[n]$
  - $T$ is $N \times N$ non-singular transformation matrix
- Then, $\mathbf{q}[n] = T^{-1}\mathbf{v}[n]$

- You and your friend could have different (valid) state variable choices for same state space representation
Solution to DT State Equations

- Two approaches
  - Time-domain solution
  - Z-transform solution
DT: Time-Domain Solution

\[ q[n + 1] = A q[n] + B x[n] \]
\[ y[n] = C q[n] + D x[n] \]

- Solve for state iteratively given an initial state \( q[0] \)

\[
q[n + 1] = A q[n] + B x[n] \\
q[1] = A q[0] + B x[0] \\
= A^2 q[0] + A B x[0] + B x[1] \\
\vdots \\
q[n] = A^n q[0] + A^{n-1} B x[0] + \ldots + B x[n - 1] \\
= A^n q[0] + \sum_{k=0}^{n-1} A^{n-1-k} B x[k] \quad n > 0
\]

- Use this to solve for the output

\[
y[n] = C q[n] + D x[n] \\
= \underbrace{C A^n q[0]}_{\text{zero-input response}} + \sum_{k=0}^{n-1} \underbrace{C A^{n-1-k} B x[k]}_{\text{zero-state response}} + D x[n]
\]
DT Z-Transform Solution

- Must use unilateral z-transform due to initial conditions

\[ q[n + 1] = Aq[n] + Bx[n] \]
\[ y[n] = Cq[n] + Dx[n] \leftrightarrow zQ_u(z) - zq[0] = AQ_u(z) + BX_u(z) \]
\[ Y_u(z) = CQ_u(z) + DX_u(z) \]
\[ Q_u(z) = \begin{bmatrix} Q_{u1}(z) \\ Q_{u2}(z) \\ \vdots \\ Q_{uN}(z) \end{bmatrix} \]

- Rearranging state equation

\[
zQ_u(z) - AQ_u(z) = zq[0] + BX(z)
\]
\[
(zI - A)Q_u(z) = zq[0] + BX(z)
\]

\[
Q_u(z) = (zI - A)^{-1}zq[0] + (zI - A)^{-1}BX_u(z)
\]

\[
Zu \uparrow
\]
\[
q[n] = Z_u^{-1} \left\{ (zI - A)^{-1}z \right\} q[0] + Z_u^{-1} \left\{ (zI - A)^{-1}BX_u(z) \right\}
\]

- Plug in for output \( y[n] \)

\[
y[n] = CZ_u^{-1} \left\{ (zI - A)^{-1}z \right\} q[0] + CZ_u^{-1} \left\{ (zI - A)^{-1}BX_u(z) \right\} + Dx[n]
\]
System Function with State Equations

- $H(z)$ is defined for zero initial conditions (initial rest or bilateral Z-transform formulation)
  - E.g. $q[0] = 0$

\[
Q_u(z) = \left( zI - A \right)^{-1} zq[0] + \left( zI - A \right)^{-1} BX_u(z)
\]

\[
Q(z) = \left( zI - A \right)^{-1} BX(z)
\]

- Solve for output

\[
Y(z) = CQ(z) + DX(z)
= C(zI - A)^{-1} BX(z) + DX(z)
= \left[ C(zI - A)^{-1} B + D \right] X(z)H(z)
\]
Stability (BIBO)

- Given $\lambda_k$ eigenvalues of system matrix $A$
  - $|\lambda_k| < 1 \ \forall k$
  - $\lambda_k$ must be distinct

- Note: when Schaum’s asks about stability they are usually talking about *asymptotically stable* ($|\lambda_k| < 1$)
DT Example Problems

In WebEx lecture

- Problem 7.23
- Problem 7.8
CT State Space Representation I

• Consider a single-input single-output (SISO) CT LTI system

\[
\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_N y(t) = x(t)
\]

• To uniquely determine a complete solution (output), requires \( N \) initial conditions – one set:

\[
y(0), y^{(1)}(0), \ldots, y^{(N-1)}(0);
\text{where } y^{(k)}(t) = \frac{d^k y(t)}{dt^k}
\]

• Define state variables (less obvious for CT)
  • Generally, output of integral block,
  • Here shortcut to \( y(t) \) derivatives because no \( x(t) \) derivatives

\[
\begin{align*}
q_1(t) &= y(t) \\
q_2(t) &= y^{(1)}(t) \\
&\vdots \\
q_N(t) &= y^{(N-1)}(t)
\end{align*}
\]
CT State Space Representation II

- Find state dot derivative (derivative “feeds” an integral block)
  - $q_k(t) = \frac{d}{dt} q_k(t)$

  $q_1(t) = y(t)$  \quad $\dot{q}_1(t) = q_2(t)$
  $q_2(t) = y^{(1)}(t)$  \quad $\dot{q}_2(t) = q_3(t)$
  \vdots  \quad \implies \quad \vdots$
  $q_N(t) = y^{(N-1)}(t)$  \quad $\dot{q}_N(t) = -a_N q_1(t) - a_{N-1} q_2(t) - \ldots - a_1 q_N(t) + x(t)$

- Note: $\frac{d^N y(t)}{dt^N} = -a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} - \ldots - a_N y(t) + x(t)$
CT State Space Representation III

- These relationships can be compactly expressed in matrix form

\[
\begin{bmatrix}
\dot{q}_1(t) \\
\dot{q}_2(t) \\
\vdots \\
\dot{q}_N(t)
\end{bmatrix}_{N \times 1} =
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
-\alpha_N & -\alpha_{N-1} & -\alpha_{N-2} & -\alpha_{N-3} & \cdots & -\alpha_1
\end{bmatrix}_{N \times N}
\begin{bmatrix}
q_1(t) \\
q_2(t) \\
\vdots \\
q_N(t)
\end{bmatrix}_{N \times 1} +
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}_{N \times 1} x(t)
\]

\[
y(t)_1 = \begin{bmatrix}
1 & 0 & \cdots & 0
\end{bmatrix}_{1 \times N}
\begin{bmatrix}
q_1(t) \\
q_2(t) \\
\vdots \\
q_N(t)
\end{bmatrix}_{N \times 1} +
\begin{bmatrix}
0
\end{bmatrix}_{1 \times 1} x(t)
\]
CT State Space Representation IV

- Generalizes for MIMO systems as

\[
\begin{bmatrix}
    \dot{q}(t) \\
    \dot{x}_1(t) \\
    \dot{x}_2(t) \\
    \vdots \\
    \dot{x}_m(t)
\end{bmatrix}_{N \times m} = \begin{bmatrix}
    A_{N \times N} q(t) \\
    B_{N \times m} \begin{bmatrix}
        x_1(t) \\
        x_2(t) \\
        \vdots \\
        x_m(t)
    \end{bmatrix}_{N \times m}
\end{bmatrix}_{N \times 1}
\]

\[
\begin{bmatrix}
    y_1(t) \\
    y_2(t) \\
    \vdots \\
    y_p(t)
\end{bmatrix}_{p \times N} = \begin{bmatrix}
    C_{p \times N} q(t) \\
    D_{p \times m} \begin{bmatrix}
        x_1(t) \\
        x_2(t) \\
        \vdots \\
        x_m(t)
    \end{bmatrix}_{m \times 1}
\end{bmatrix}_{p \times 1}
\]
CT Laplace Transform Solution

• Must use unilateral LT due to initial conditions

\[ \dot{q}(t) = Aq(t) + Bx(t) \quad \Longleftrightarrow \quad sQ_u(s) - q(0) = AQ_u(s) + BX_u(s) \]

\[ y(t) = Cq(t) + Dx(t) \quad \Rightarrow \quad Y_u(s) = CQ_u(s) + DX_u(s) \]

• Rearranging state equation

\[ sQ_u(s) - AQ_u(s) = q(0) + BX_u(s) \]

\[ (sI - A)Q_u(s) = q(0) + BX_u(s) \]

\[ \Rightarrow Q_u(s) = (sI - A)^{-1}q(0) + (sI - A)^{-1}BX_u(s) \]

• Plug in for output

\[ Y(s) = C\left[(sI - A)^{-1}q(0)\right] + C(sI - A)^{-1}BX_u(s) + DX_u(s) \]

\[ = C(sI - A)^{-1}q(0) + [C(sI - A)^{-1}B + D]X_u(s) \]

zero-input response \quad \text{zero-state response}

\[ L_u \updownarrow \]

\[ y(t) = y_{zir}(t) + y_{zsr}(t) \]
Determining System Function

- From previous example
  - \[ H(s) = \frac{Y(s)}{X(s)} = c(sI - A)^{-1}b + d \]

- When MIMO
  - \[ H(s) = C(sI - A)^{-1}B + D \]
    - Each element \( H_{ij}(s) \) of \( H(s) \) matrix is the transfer function relating output \( y_i(t) \) to input \( x_j(t) \)
Stability (BIBO)

• Given $\lambda_k$ eigenvalues of system matrix $A$
  1. $\text{Re}\{\lambda_k\} < 0$ $\forall k$
  2. $\lambda_k$ must be distinct

• Note: when Schaum’s asks about stability they are usually talking about *asymptotically stable* ($\text{Re}\{\lambda_k\} < 0$)
CT Example Problems

- Problem 7.48

In WebEx lecture