Homework #9
Due Th 5/07

A number of these homework problems require you first go through the "Solved Problems" since the description/definition is not in the chapter material.

You are allowed to use Matlab (or similar) to help solve these problems but will be required to know how to do them by hand for the Final Exam. As an example, you may want to find the inverse using the Symbolic Toolbox:

```
syms z;                     % create symbolic variable z
A = eye(3);                 % create simple system matrix
G = (z*eye(3) -A)^-1        % find inverse
```

Other Matlab functions that may be helpful include `inv.m, rank.m, eig.m`.

1. (Schaum 7.6) **[half points]**

   Note this problem is solved in the book.

   Solution

   See book solutions. Use the library [link] for remote access off campus.

2. (Schaum 7.16 - 7.17) **[half points]**

   Note this problem is solved in the book already but highlights the difference between canonical forms.

   Solution

   See book solutions. Use the library [link] for remote access off campus.

3. (Schaum 7.65)

   Solution

   Notice this is a multi-output problem.

   The outputs can be found by inspection as:

$$
\begin{aligned}
y_1(t) &= \frac{x(t) - v_C(t)}{R_1} = -\frac{1}{R_1}q_2(t) + \frac{1}{R_1}x(t) = -q_2(t) + x(t) \\
y_2(t) &= q_2(t)
\end{aligned}
$$

   The state equations require more effort. First, do KVL on the right loop:

$$
\begin{aligned}
q_1(t) &= i_L(t) \\
\Rightarrow L\frac{di_L(t)}{dt} + i_L(t)R &= q_2(t) \\
\frac{q_1(t)}{dt} &= -\frac{R}{L}q_1(t) + q_2(t)
\end{aligned}
$$

   Next, KCL @ the node above the inductor:

$$
\begin{aligned}
q_2(t) &= v_c(t) \\
\Rightarrow i_c(t) + i_L(t) &= \frac{x(t) - v_c(t)}{R} \\
\frac{dv_c(t)}{dt} + q_1(t) &= \frac{x(t) - q_2(t)}{R} \\
\frac{dq_2(t)}{dt} &= -q_1(t) - \frac{1}{R}q_2(t) + x(t)
\end{aligned}
$$

This results in the state space equations

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t)$$
$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t)$$

4. (Schaum 7.68)

Solution

(a) Using the states as labeled,

$$\dot{q}_1(t) = -2q_1(t) - 3q_2(t) \qquad\qquad y(t) = q_1(t) + q_2(t)$$
$$\dot{q}_2(t) = q_2(t) + x(t)$$

These relationships result in state-space representation,

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} -2 & -3 \\ 0 & 1 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t) \qquad\qquad y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{q}(t) + 0x(t)$$

(b) The eigenvalues of matrix A can be found using Matlab's `eig.m` function. The eigenvalues are $\lambda = \begin{bmatrix} -2 & 1 \end{bmatrix}^T$. Since the real part of $Re\{\lambda_2\} > 0$, this is not an asymptotically stable system.

(c)

$$H(s) = c(sI - A)^{-1}b$$

Define $G = (sI - A)^{-1}$

$$\begin{aligned} G &= \begin{bmatrix} s+2 & 3 \\ 0 & s-1 \end{bmatrix}^{-1} \\ &= \frac{1}{(s+2)(s-1)} \begin{bmatrix} s-1 & -3 \\ 0 & s+2 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ H(s) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = b + d \\ &= \frac{-3 + s + 2}{(s+2)(s-1)} = \frac{s-1}{(s+2)(s-1)} \\ &= \frac{1}{s+2} \end{aligned}$$

(d) Since the system only has a pole @ $s = -2$ and is causal, this implies the system is BIBO stable. Notice, that even though part (b) was unstable, because of pole/zero cancellation, this turns out to be a BIBO stable system. Therefore it is important to check the system function to determine stability.

5. (Schaum 7.71(b))

Solution

$$\begin{aligned}
e^{At} &= \mathcal{L}_u^{-1}\left\{(sI-A)^{-1}\right\} \\
&= \mathcal{L}_u^{-1}\left\{\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}\right\} = \mathcal{L}_u^{-1}\left\{\frac{1}{s(s+3)+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\right\} \\
&= \mathcal{L}_u^{-1}\left\{\frac{1}{(s+2)(s+1)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\right\} \\
&= \mathcal{L}_u^{-1}\left\{\begin{bmatrix} \frac{A}{s+2}+\frac{B}{s+1} & \frac{C}{s+2}+\frac{D}{s+1} \\ \frac{E}{s+2}+\frac{F}{s+1} & \frac{G}{s+2}+\frac{H}{s+1} \end{bmatrix}\right\}
\end{aligned}$$

Doing PFE on each element of the 2×2 matrix

$$A = \left.\frac{s+3}{s+1}\right|_{s=-2} = \frac{1}{-1} = -1 \qquad B = \left.\frac{s+3}{s+2}\right|_{s=-1} = 2$$

$$C = \left.\frac{1}{s+1}\right|_{s=-2} = -1 \qquad D = \left.\frac{1}{s+2}\right|_{s=-1} = 1$$

$$E = \left.\frac{-2}{s+1}\right|_{s=-2} = 2 \qquad F = \left.\frac{-2}{s+2}\right|_{s=-1} = -2$$

$$G = \left.\frac{s}{s+1}\right|_{s=-2} = 2 \qquad H = \left.\frac{s}{s+2}\right|_{s=-1} = -1$$

Resulting in matrix elements that have the inverse Laplace transform taken by inspection

$$\begin{aligned}
e^{At} &= \mathcal{L}_u^{-1}\left\{\begin{bmatrix} \frac{-1}{s+2}+\frac{2}{s+1} & \frac{-1}{s+2}+\frac{1}{s+1} \\ \frac{2}{s+2}+\frac{-2}{s+1} & \frac{2}{s+2}+\frac{-1}{s+1} \end{bmatrix}\right\} \\
&= \begin{bmatrix} -e^{-2t}+2e^{-t} & -e^{-2t}+e^{-t} \\ 2e^{-2t}-2e^{-t} & 2e^{-2t}-e^{-t} \end{bmatrix} u(t)
\end{aligned}$$

6. (Schaum 7.72)

Please solve it without trying to use the $\left[e^{-t}\cos t\right]u(t)$ types of terms but instead just using normal PFE. Note this means you will have complex α in the $e^{-\alpha t}u(t)$ terms.

Solution

Notice that you will be using the system from 7.65 to find the output with the equation

$$Y(s) = C(sI-A)^{-1}\mathbf{q}[0] + [C(sI-A)^{-1}b + d]X(s).$$

Since we are told to solve this under initially relaxed conditions, the output only depends on the input $x(t) = u(t) \leftrightarrow X(s) = \frac{1}{s}$

$$Y(s) = [C(sI-A)^{-1}b + d]X(s).$$

To solve this, define

$$G = (sI-A)^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and note that since we only need to find $v_c(t) = y_2(t)$ only the bottom row of the output equations (state space matrices C and d) is required.

$$\begin{aligned} Y(s) &= [c(sI-A)^{-1}b + d]X(s) \\ &= \left[\begin{bmatrix}0 & 1\end{bmatrix} G \begin{bmatrix}0\\1\end{bmatrix} + 0\right]\frac{1}{s} \\ &= \left[\begin{bmatrix}0 & 1\end{bmatrix} \begin{bmatrix}a & b\\ c & d\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}\right]\frac{1}{s} \\ &= \left[\begin{bmatrix}c & d\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}\right]\frac{1}{s} \\ &= d\frac{1}{s} \end{aligned}$$

The d entry of G can be found by taking the inverse

$$\begin{aligned} G = (sI-A)^{-1} &= \begin{bmatrix}s+1 & -1\\ 1 & s+1\end{bmatrix}^{-1} \\ &= \frac{1}{(s+1)^2+1}\begin{bmatrix}s+1 & 1\\ -1 & s+1\end{bmatrix} \\ \Rightarrow d &= \frac{s+1}{s^2+2s+2}. \end{aligned}$$

Finally, the output can be found by PFE and inverse LT,

$$\begin{aligned} Y(s) &= d\frac{1}{s} \\ &= \frac{s+1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{B}{s+(1-j)} + \frac{C}{s+(1+j)} \\ &= \frac{1/2}{s} + \frac{1/(2(j-1))}{s+(1-j)} + \frac{-1/(2(j+1))}{s+(1+j)} \end{aligned}$$

The output can be found using a Laplace Transform table as

$$y(t) = \frac{1}{2}u(t) + \frac{1}{2(j-1)}e^{-(1-j)t}u(t) - \frac{1}{2(j+1)}e^{-(1+j)t}u(t).$$

7. (Schaum 7.73)

Solution

See problem 7.48 for a similar problem.

Since, there are no differentials on $x(t)$, let the state variables be directly related to the output

$$q_1(t) = y(t) \qquad\qquad q_2(t) = y'(t)$$

This results in the following relationships

$$\begin{aligned} \dot{q}_1(t) &= q_2(t) \\ \dot{q}_2(t) &= -3q_2(t) - 2q_1(t) \\ y(t) &= q_1(t) \end{aligned}$$

Arranging into matrix form the state equations are

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{q}(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{q}(t)$$

The output with the given initial conditions can be found as

$$y(t) = ce^{At}\mathbf{q}(0) + \int_0^t ce^{A(t-\tau)} bx(\tau)d\tau$$

Using the result from (7.71) for e^{At},

$$e^{At} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$ce^{At}\mathbf{q}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b$$
$$= (e^{-t} - e^{-2t})u(t)$$
$$ce^{A(t-\tau)}b = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

The final result is just from the initial conditions (since there is no input $x(t)$)

$$y(t) = (e^{-t} - e^{-2t})u(t)$$