Homework #8
Due Th 4/30

A number of these homework problems require you first go through the "Solved Problems" since the description/definition is not in the chapter material.

You are allowed to use Matlab (or similar) to help solve these problems but will be required to know how to do them by hand for the Final Exam. As an example, you may want to find the inverse using the Symbolic Toolbox:

```
syms z;                     % create symbolic variable z
A = eye(3);                 % create simple system matrix
G = (z*eye(3) -A)^-1        % find inverse
```

Other Matlab functions that may be helpful include `inv.m, rank.m, eig.m`.

Remember the inverse of a $2 \times 2$ matrix can be found as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

1. (Schaum 7.9 - 7.10)

   Note this problem is solved in the book already but highlights the difference between Direct Form II (Fig 7-9 Canonical simulation of the second form) and Direct Form II Transposed (Fig. 7-8 Canonical simulation of the first form). Be sure to understand the difference between the two of these forms.

   **Solution**

   See book solutions. Use the library [link] for remote access off campus.

2. (Schaum 7.56)

   **Solution**

   It is easiest to start with the output equations since the output signal is tied to the output of a delay element.

$$\begin{aligned} y_1[n] &= q_1[n] \\ y_2[n] &= q_2[n] \end{aligned}$$

   Putting these together into matrix form

$$\underbrace{\begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix}}_{\mathbf{y}[n]} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}}_{\mathbf{q}[n]} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{d} x[n]$$

   Similarly, next state equations can be defined by tracing through the block diagrams.

$$\begin{aligned} q_1[n+1] &= -\frac{1}{3} y_1[n] + x[n] = -\frac{1}{3} q_1[n] + x[n] \\ q_2[n+1] &= x[n] + \frac{1}{3} y_1[n] - \frac{1}{2} y_2[n] \\ &= x[n] + \frac{1}{3} q_1[n] - \frac{1}{2} q_2[n] \end{aligned}$$

This results in the matrix form of

$$\mathbf{q}[n+1] = \underbrace{\begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}}_{A} \mathbf{q}[n] + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b} x[n]$$

3. (Schaum 7.57)

**Solution**

(a) Define the node above the top delay (feeding $q_1[n]$)

$$w[n] = x[n] + q_1[n] - \frac{1}{2}q_2[n].$$

The state space representation can be found as follows:
Measurement equation:

$$\begin{aligned} y[n] &= \frac{1}{3}x[n] + q_1[n] - \frac{1}{2}w[n] \\ &= \frac{1}{3}x[n] + q_1[n] - \frac{1}{2}(x[n] + q_1[n] - \frac{1}{2}q_2[n]) \\ &= -\frac{1}{6}x[n] + \frac{1}{2}q_1[n] + \frac{1}{4}q_2[n] \\ y[n] &= \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix}}_{c} \underbrace{\begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}}_{\mathbf{q}[n]} + \underbrace{\begin{bmatrix} -\frac{1}{6} \end{bmatrix}}_{d} x[n] \end{aligned}$$

State equation:

$$\begin{aligned} q_1[n+1] &= w[n] = x[n] + q_1[n] - \frac{1}{2}q_2[n] \\ q_2[n+1] &= q_1[n] \\ \mathbf{q}[n+1] &= \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}}_{A} \mathbf{q}[n] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b} x[n] \end{aligned}$$

(b) The system function is found using the following equation

$$H(z) = \left[c(zI - A)^{-1}b + d\right]$$

Define $G = (zI - A)^{-1}$

$$
\begin{aligned}
G &= \begin{bmatrix} z-1 & \frac{1}{2} \\ -1 & z \end{bmatrix}^{-1} \\
&= \frac{1}{(z-1)z + \frac{1}{2}} \begin{bmatrix} z & -\frac{1}{2} \\ 1 & z-1 \end{bmatrix} \\
&= \frac{1}{z^2 - z + \frac{1}{2}} \begin{bmatrix} z & -\frac{1}{2} \\ 1 & z-1 \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
H(z) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{6} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} - \frac{1}{6} \\
&= \frac{1}{2}a + \frac{1}{4}c - \frac{1}{6}
\end{aligned}
$$

Using Matlab and $G$ above,

$$
\begin{aligned}
H(z) &= \frac{z/2}{z^2 - z + \frac{1}{2}} + \frac{1/4}{z^2 - z + \frac{1}{2}} - \frac{1/6(z^2 - z + \frac{1}{2})}{z^2 - z + \frac{1}{2}} \\
&= -\frac{1}{6} \frac{z^2 - 4z - 1}{z^2 - z + \frac{1}{2}}
\end{aligned}
$$

(c) Using part (b) multiplied by $\frac{z^{-2}}{z^{-2}}$,

$$
H(z) = \frac{Y(z)}{X(z)}
$$

$$
y[n] - y[n-1] + \frac{1}{2}y[n-2] = -\frac{1}{6}\left(x[n] - 4x[n-1] - x[n-2]\right)
$$

4. (Schaum 7.58)

You only need to provide one of the canonical forms. Also, draw the block diagram for the form you select.

**Solution**

Recognize coefficients from difference equation

$$
\begin{aligned}
a_1 &= 1 & b_0 &= 0 \\
a_2 &= -6 & b_1 &= 2 \\
& & b_2 &= 1
\end{aligned}
$$

Using result eq (7.91) from problem 7.10,

$$
\mathbf{v}[n+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix}}_{A} \mathbf{v}[n] + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} x[n]
$$

$$
y[n] = \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{c} \mathbf{v}[n] + \underbrace{[0]}_{d} x[n]
$$

Using result eq (7.91) from problem 7.9,

$$\mathbf{q}[n+1] = \underbrace{\begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}}_{A} \mathbf{v}[n] + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{b} x[n]$$

$$y[n] = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{c} \mathbf{v}[n] + \underbrace{[0]}_{d} x[n]$$

5. (Schaum 7.60(b))

**Solution**

$$A^n = \mathcal{Z}^{-1}\left\{ \underbrace{(zI - A)^{-1}}_{G} z \right\}$$

Using Matlab,

$$G = \begin{bmatrix} z-3 & 0 & 0 \\ 0 & z+2 & -1 \\ 0 & -4 & z-1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{1}{z-3} & 0 & 0 \\ 0 & \frac{z-1}{z^2+z-6} & \frac{1}{z^2+z-6} \\ 0 & \frac{4}{z^2+z-6} & \frac{z+2}{z^2+z-6} \end{bmatrix}$$

$$A^n = \mathcal{Z}^{-1}\left\{ \begin{bmatrix} \frac{z}{z-3} & 0 & 0 \\ 0 & \frac{z(z-1)}{(z+3)(z-2)} & \frac{z}{(z+3)(z-2)} \\ 0 & \frac{4z}{(z+3)(z-2)} & \frac{z(z+2)}{(z+3)(z-2)} \end{bmatrix} \right\}$$

The inverse can be found by first converting to $z^{-k}$ form.

$$= \mathcal{Z}^{-1}\left\{ \begin{bmatrix} \frac{1}{1-3z^{-1}} & 0 & 0 \\ 0 & \frac{1-z^{-1}}{(1+3z^{-1})(1-2z^{-1})} & \frac{z^{-1}}{(1+3z^{-1})(1-2z^{-1})} \\ 0 & \frac{4z^{-1}}{(1+3z^{-1})(1-2z^{-1})} & \frac{1+2z^{-1}}{(1+3z^{-1})(1-2z^{-1})} \end{bmatrix} \right\}$$

Perform PFE on each matrix element

$$= \mathcal{Z}^{-1}\left\{ \begin{bmatrix} \frac{1}{1-3z^{-1}} & 0 & 0 \\ 0 & \frac{A}{1+3z^{-1}} + \frac{B}{1-2z^{-1}} & \frac{C}{1+3z^{-1}} + \frac{D}{1-2z^{-1}} \\ 0 & \frac{E}{1+3z^{-1}} + \frac{F}{1-2z^{-1}} & \frac{G}{1+3z^{-1}} + \frac{H}{1-2z^{-1}} \end{bmatrix} \right\}$$

$$A = \left.\frac{1-z^{-1}}{1-2z^{-1}}\right|_{z=-3} = \frac{1+1/3}{1+2/3} = \frac{4/3}{5/3} = \frac{4}{5} \qquad B = \left.\frac{1-z^{-1}}{1+3z^{-1}}\right|_{z=2} = \frac{1-1/2}{1+3/2} = \frac{1/2}{5/2} = \frac{1}{5}$$

$$C = \left.\frac{z^{-1}}{1-2z^{-1}}\right|_{z=-3} = \frac{-1/3}{5/3} = -\frac{1}{5} \qquad D = \left.\frac{z^{-1}}{1+3z^{-1}}\right|_{z=2} = \frac{1/2}{5/2} = \frac{1}{5}$$

$$E = \left.\frac{4z^{-1}}{1-2z^{-1}}\right|_{z=-3} = \frac{-4/3}{5/3} = -\frac{4}{5} \qquad F = \left.\frac{4z^{-1}}{1+3z^{-1}}\right|_{z=2} = \frac{2}{5/2} = \frac{4}{5}$$

$$G = \left.\frac{1+2z^{-1}}{1-2z^{-1}}\right|_{z=-3} = \frac{1-2/3}{5/3} = \frac{1}{5} \qquad H = \left.\frac{1+2z^{-1}}{1+3z^{-1}}\right|_{z=2} = \frac{1+1}{5/2} = \frac{4}{5}$$

Taking the inverse of each element results in

$$A^n = \begin{bmatrix} 3^n & 0 & 0 \\ 0 & \frac{4}{5}(-3)^n + \frac{1}{5}2^n & -\frac{1}{5}(-3)^n + \frac{1}{5}2^n \\ 0 & -\frac{4}{5}(-3)^n + \frac{4}{5}2^n & \frac{1}{5}(-3)^n + \frac{4}{5}2^n \end{bmatrix} u[n].$$

6. (Schaum 7.62)

**Solution**

(a)

$$H(z) = \begin{bmatrix} c(zI - A)^{-1}b + d \end{bmatrix}$$

Define $G = (zI - A)^{-1}$

$$G = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{aligned} H(z) &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a+c \\ d+f \\ g+i \end{bmatrix} \\ &= d + f \end{aligned}$$

Using Matlab,

$$\begin{aligned} H(z) &= 0 + \frac{1}{z^2 - 2z + 1} = \frac{z^{-2}}{(1 - z^{-1})(1 - z^{-1})} = \frac{A}{z-1} + \frac{B}{(z-1)^2} \\ B &= 1 \qquad A = 0 \\ &= \frac{1}{(z-1)^2} \end{aligned}$$

(b) Using results from problem 7.33,

$$\begin{aligned} M_c &= \begin{bmatrix} b & Ab & A^2b \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \end{aligned}$$

Using Matlab to find the rank (`rank(Mc)`) is 3 (full rank), this system is controllable.

(c) Using results from problem 7.34,

$$\begin{aligned} M_o &= \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \end{aligned}$$

Using Matlab to find the rank (`rank(Mo)`) is 2 (not full rank), this system is not observable.