Homework #8
Due Tu. 5/02

A number of these homework problems require you first go through the “Solved Problems” since the description/definition is not in the chapter material.

You are allowed to use Matlab (or similar) to help solve these problems but will be required to know how to do them by hand for the Final Exam. As an example, you may want to find the inverse using the Symbolic Toolbox:

```matlab
syms z; % create symbolic variable z
A = eye(3); % create simple system matrix
G = (z*eye(3) -A)^-1 % find inverse
```

Other Matlab functions that may be helpful include `inv.m`, `rank.m`, `eig.m`.

Remember the inverse of a 2×2 matrix can be found as

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\]

1. (Schaum 7.57)

**Solution**

(a) Define the node above the top delay (feeding \( q_1[n] \))

\[
w[n] = x[n] + q_1[n] - \frac{1}{2} q_2[n].
\]

The state space representation can be found as follows:

**Measurement equation:**

\[
y[n] = \frac{1}{3} x[n] + q_1[n] - \frac{1}{2} w[n]
\]

\[
= \frac{1}{3} x[n] + q_1[n] - \frac{1}{2} (x[n] + q_1[n] - \frac{1}{2} q_2[n])
\]

\[
= -\frac{1}{6} x[n] + \frac{1}{2} q_1[n] + \frac{1}{4} q_2[n]
\]

\[
y[n] = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ c \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ q_1[n] \end{bmatrix} + \begin{bmatrix} -\frac{1}{6} \\ 0 \\ d \end{bmatrix} x[n]
\]

**State equation:**

\[
q_1[n + 1] = w[n] = x[n] + q_1[n] - \frac{1}{2} q_2[n]
\]

\[
q_2[n + 1] = q_1[n]
\]

\[
q[n + 1] = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} q[n] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n]
\]
(b) The system function is found using the following equation

\[ H(z) = [c(zI - A)^{-1}b + d] \]

Define \( G = (zI - A)^{-1} \)

\[
G = \begin{bmatrix} z - \frac{1}{2} & 1 \\ -1 & z \end{bmatrix}^{-1} \\
= \frac{1}{(z-1)z + \frac{1}{2}} \begin{bmatrix} z & -\frac{1}{2} \\ 1 & z - 1 \end{bmatrix} \\
= \frac{1}{z^2 - z + \frac{1}{2}} \begin{bmatrix} z & -\frac{1}{2} \\ 1 & z - 1 \end{bmatrix} \\
= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
H(z) = [\frac{1}{2} \ 1 \ 4] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{6} \\
= [\frac{1}{2} \ 1 \ 4] \begin{bmatrix} a \\ c \end{bmatrix} - \frac{1}{6} \\
= \frac{1}{2}a + \frac{1}{4}c - \frac{1}{6}
\]

Using Matlab and \( G \) above,

\[
H(z) = \frac{z/2}{z^2 - z + \frac{1}{2}} + \frac{1/2}{z^2 - z + \frac{1}{2}} - \frac{1/6(z^2 - z + \frac{1}{2})}{z^2 - z + \frac{1}{2}} \\
= -\frac{1}{6} \frac{z^2 - 4z - 1}{z^2 - z + \frac{1}{2}}
\]

(c) Using part (b) multiplied by \( \frac{z^2 - 2}{z-1} \),

\[
H(z) = \frac{Y(z)}{X(z)} \\
y[n] - y[n - 1] + \frac{1}{2}y[n - 2] = -\frac{1}{6} (x[n] - 4x[n - 1] - x[n - 2])
\]

2. (Schaum 7.58)

You only need to provide one of the canonical forms. Also, draw the block diagram for the form you select.

**Solution**

Recognize coefficients from difference equation

\[
a_1 = 1 \quad b_0 = 0 \\
a_2 = -6 \quad b_1 = 2 \\
b_2 = 1
\]
Using result eq (7.91) from problem 7.10,

\[ \mathbf{v}[n+1] = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} \mathbf{v}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n] \]

\[ y[n] = \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{v}[n] + \begin{bmatrix} 0 \end{bmatrix} x[n] \]

Using result eq (7.91) from problem 7.9,

\[ \mathbf{q}[n+1] = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \mathbf{v}[n] + \begin{bmatrix} 2 \\ 1 \end{bmatrix} x[n] \]

\[ y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{v}[n] + \begin{bmatrix} 0 \end{bmatrix} x[n] \]

3. (Schaum 7.60(b))

Solution

\[ A^n = Z^{-1} \left\{ \left( zI - A \right)^{-1} z \right\} \]

Using Matlab,

\[ G = \begin{bmatrix} z - 3 & 0 & 0 \\ 0 & z + 2 & -1 \\ 0 & -4 & z - 1 \end{bmatrix}^{-1} \]

\[ = \begin{bmatrix} \frac{1}{z-3} & 0 & 0 \\ 0 & \frac{z-1}{z-3} & \frac{1}{z-3} \\ 0 & \frac{z-1}{z^2+z-6} & \frac{z}{z^2+z-6} \end{bmatrix} \]

\[ A^n = Z^{-1} \left\{ \begin{bmatrix} \frac{z}{z-3} & 0 & 0 \\ 0 & \frac{z(z-1)}{(z+3)(z-2)} & \frac{z}{(z+3)(z-2)} \\ 0 & \frac{z}{(z+3)(z-2)} & \frac{z(z+2)}{(z+3)(z-2)} \end{bmatrix} \right\} \]

The inverse can be found by first converting to \( z^{-k} \) form.

\[ = Z^{-1} \left\{ \begin{bmatrix} \frac{1}{1-3z^{-1}} & 0 & 0 \\ 0 & \frac{1-z^{-1}}{(1+3z^{-1})(1-2z^{-1})} & \frac{z^{-1}}{1+3z^{-1}} \\ 0 & \frac{1}{1+3z^{-1}} & \frac{1-z^{-1}}{(1+3z^{-1})(1-2z^{-1})} \end{bmatrix} \right\} \]

Perform PFE on each matrix element

\[ = Z^{-1} \left\{ \begin{bmatrix} \frac{1}{1-3z^{-1}} & 0 & 0 \\ 0 & \frac{A}{1+3z^{-1}} + \frac{B}{1-2z^{-1}} & \frac{C}{1+3z^{-1}} + \frac{D}{1-2z^{-1}} \\ 0 & \frac{E}{1+3z^{-1}} + \frac{F}{1-2z^{-1}} & \frac{G}{1+3z^{-1}} + \frac{H}{1-2z^{-1}} \end{bmatrix} \right\} \]
Taking the inverse of each element results in

\[
A^n = \begin{bmatrix}
3^n & 0 & 0 \\
0 & \frac{4}{3}(-3)^n + \frac{1}{5}2^n & -\frac{1}{5}(-3)^n + \frac{1}{5}2^n \\
0 & -\frac{4}{5}(-3)^n + \frac{4}{5}2^n & \frac{1}{5}(-3)^n + \frac{4}{5}2^n
\end{bmatrix} u[n]
\]

4. (Schaum 7.62)

Solution

(a) 

\[H(z) = [c(zI - A)^{-1}b + d]\]

Define \(G = (zI - A)^{-1}\)

\[
G = \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\]

\[H(z) = \begin{bmatrix}0 & 1 & 0\end{bmatrix} \begin{bmatrix}a & b & c \\
d & e & f \\
g & h & i\end{bmatrix} \begin{bmatrix}1 \\
0 \\
1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\end{bmatrix} \begin{bmatrix}a + c \\
d + f \\
g + i\end{bmatrix} = d + f
\]

Using Matlab,

\[H(z) = 0 + \frac{1}{z^2 - 2z + 1} = \frac{z^{-2}}{(1 - z^{-1})(1 - z^{-1})} = \frac{A}{z - 1} + \frac{B}{(z - 1)^2}
\]

\[B = 1 \quad A = 0
\]

\[= \frac{1}{(z - 1)^2}
\]

(b) Using results from problem 7.33,

\[M_c = \begin{bmatrix}b & Ab & A^2b\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 \\
0 & 1 & 2 \\
1 & 2 & 3\end{bmatrix}
\]

Using Matlab to find the rank (\(\text{rank}(M_c)\)) is 3 (full rank), this system is controllable.
(c) Using results from problem 7.34,

\[ M_o = \begin{bmatrix}
    c & cA \\
    cA & cA^2 \\
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    0 & -1 & 2
\end{bmatrix} \]

Using Matlab to find the rank \( \text{rank}(M_o) \) is 2 (not full rank), this system is not observable.

5. (Schaum 7.65)

**Solution**

Notice this is a multi-output problem.

The outputs can be found by inspection as:

\[
y_1(t) = \frac{1}{R_1} q_2(t) + \frac{1}{R_1} x(t) = q_2(t) + x(t) \\
y_2(t) = q_2(t)
\]

The state equations require more effort. First, do KVL on the right loop:

\[
q_1(t) = i_L(t) \\
\Rightarrow L \frac{di_L(t)}{dt} + i_L(t)R = q_2(t) \\
\frac{q_1(t)}{dt} = -\frac{R}{L} q_1(t) + q_2(t)
\]

Next, KCL @ the node above the inductor:

\[
q_2(t) = v_c(t) \\
\Rightarrow i_c(t) + i_L(t) = \frac{x(t) - v_c(t)}{R} \\
dv_c(t) + q_1(t) = \frac{x(t) - q_2(t)}{R} \\
\frac{dq_2(t)}{dt} = -q_1(t) - \frac{1}{R} q_2(t) + x(t)
\]

This results in the state space equations

\[
\dot{q}(t) = \begin{bmatrix}
    \dot{q}_1(t) \\
    \dot{q}_2(t)
\end{bmatrix} = \begin{bmatrix}
    -1 & 1 \\
    -1 & -1
\end{bmatrix} q(t) + \begin{bmatrix}
    0 \\
    1
\end{bmatrix} x(t)
\]

\[
y(t) = \begin{bmatrix}
    y_1(t) \\
    y_2(t)
\end{bmatrix} = \begin{bmatrix}
    0 & -1 \\
    0 & 1
\end{bmatrix} q(t) + \begin{bmatrix}
    1 \\
    0
\end{bmatrix} x(t)
\]
6. (Schaum 7.68)

Solution

(a) Using the states as labeled,

\[ \dot{q}_1(t) = -2q_1(t) - 3q_2(t) \]
\[ \dot{q}_2(t) = q_2(t) + x(t) \]
\[ y(t) = q_1(t) + q_2(t) \]

These relationships result in state-space representation,

\[ \dot{\mathbf{q}}(t) = \begin{bmatrix} -2 & -3 \\ 0 & 1 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t) \]
\[ y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{q}(t) + 0x(t) \]

(b) The eigenvalues of matrix $A$ can be found using Matlab’s `eig.m` function. The eigenvalues are $\lambda = [-2 \ 1]^T$. Since the real part of $Re\{\lambda_2\} > 0$, this is not an asymptotically stable system.

(c)

\[ H(s) = c(sI - A)^{-1}b \]

Define $G = (sI - A)^{-1}$

\[ G = \begin{bmatrix} s + 2 & 3 \\ 0 & s - 1 \end{bmatrix}^{-1} \]
\[ = \frac{1}{(s + 2)(s - 1)} \begin{bmatrix} s - 1 & -3 \\ 0 & s + 2 \end{bmatrix} \]
\[ = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

\[ H(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = b + d \]
\[ = \frac{-3 + s + 2}{(s + 2)(s - 1)} = \frac{s - 1}{(s + 2)(s - 1)} \]
\[ = \frac{1}{s + 2} \]

(d) Since the system only has a pole @ $s = -2$ and is causal, this implies the system is BIBO stable. Notice, that even though part (b) was unstable, because of pole/zero cancellation, this turns out to be a BIBO stable system. Therefore it is important to check the system function to determine stability.
7. (Schaum 7.71(b))

**Solution**

\[ e^{At} = \mathcal{L}^{-1}_u \{(sI - A)^{-1}\} \]

\[ = \mathcal{L}^{-1}_u \left\{ \begin{bmatrix} s & -1 \\ 2 & s + 3 \end{bmatrix} \right\} = \mathcal{L}^{-1}_u \left\{ \frac{1}{s(s + 3) + 2} \begin{bmatrix} s + 3 & 1 \\ -2 & s \end{bmatrix} \right\} \]

\[ = \mathcal{L}^{-1}_u \left\{ \frac{1}{(s + 2)(s + 1)} \begin{bmatrix} s + 3 & 1 \\ -2 & s \end{bmatrix} \right\} \]

\[ = \mathcal{L}^{-1}_u \left\{ \frac{A}{s + 2} + \frac{B}{s + 1} \frac{C}{s + 2} + \frac{D}{s + 1} \frac{E}{s + 2} + \frac{F}{s + 1} \frac{G}{s + 2} + \frac{H}{s + 1} \right\} \]

Doing PFE on each element of the 2 × 2 matrix

\[
A = \frac{s + 3}{s + 1} \bigg|_{s = -2} = \frac{1}{-1} = -1 \\
B = \frac{s + 3}{s + 2} \bigg|_{s = -1} = 2 \\
C = \frac{1}{s + 1} \bigg|_{s = -2} = -1 \\
D = \frac{1}{s + 2} \bigg|_{s = -1} = 1 \\
E = \frac{-2}{s + 1} \bigg|_{s = -2} = 2 \\
F = \frac{-2}{s + 2} \bigg|_{s = -1} = -2 \\
G = \frac{s}{s + 1} \bigg|_{s = -2} = 2 \\
H = \frac{s}{s + 2} \bigg|_{s = -1} = -1
\]

Resulting in matrix elements that have the inverse Laplace transform taken by inspection

\[ e^{At} = \mathcal{L}^{-1}_u \left\{ \begin{bmatrix} \frac{-1}{s + 2} + \frac{2}{s + 1} & \frac{-1}{s + 2} + \frac{1}{s + 1} \\ \frac{1}{s + 2} + \frac{2}{s + 1} & \frac{1}{s + 2} + \frac{1}{s + 1} \end{bmatrix} \right\} \]

\[ = \begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix} u(t) \]

8. (Schaum 7.73)

**Solution**

See problem 7.48 for a similar problem.

Since, there are no differentials on \(x(t)\), let the state variables be directly related to the output

\[ q_1(t) = y(t) \quad q_2(t) = y'(t) \]

This results in the following relationships

\[ \dot{q}_1(t) = q_2(t) \]
\[ \dot{q}_2(t) = -3q_2(t) - 2q_1(t) \]
\[ y(t) = q_1(t) \]

Arranging into matrix form the state equations are

\[ \dot{\mathbf{q}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{q}(t) \]
\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{q}(t) \]
The output with the given initial conditions can be found as

\[ y(t) = ce^{At}q(0) + \int_0^t ce^{A(t-\tau)}bx(\tau)d\tau \]

Using the result from (7.71) for \( e^{At} \),

\[ e^{At} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

\[ ce^{At}q(0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b \]

\[ = (e^{-t} - e^{-2t})u(t) \]

\[ ce^{A(t-\tau)}b = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \]

The final result is just from the initial conditions (since there is no input \( x(t) \))

\[ y(t) = (e^{-t} - e^{-2t})u(t) \]