Homework #6
Due Su. 3/31

Note: The **Basic Problems with Answers** will be worth half as much as the other questions. You must show all your work to receive credit.

1. (OW 9.29)

Solution

(a)

$$x(t) = e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1} \qquad \mathcal{R}e\{s\} > -1$$
$$h(t) = e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2} \qquad \mathcal{R}e\{s\} > -2$$

(b)

$$Y(s) = X(s)H(s) = \frac{1}{(s+1)(s+2)} \qquad \mathcal{R}e\{s\} > -1$$

(c)

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2}$$
$$A = \left.\frac{1}{s+2}\right|_{s=-1} = 1$$
$$B = \left.\frac{1}{s+1}\right|_{s=-2} = -1$$
$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$
$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

(d)

$$y(t) = x(t) * y(t) = \int_0^t e^{-2\tau}e^{-(t-\tau)}d\tau \qquad t > 0$$
$$= e^{-t}\int_0^t e^{-\tau}d\tau = e^{-t}\left[-e^{-\tau}\right]_0^t = e^{-t}[1 - e^{-t}]$$
$$= [e^{-t} - e^{-2t}]u(t)$$

2. (OW 9.30)

Solution

$$x(t) = u(t) \longleftrightarrow s(t) = (1 - e^{-t} - te{-}t)u(t)$$
$$X(s) = \frac{1}{s} \qquad \mathcal{R}e\{s\} > 0$$
$$S(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} \qquad \mathcal{R}e\{s\} > 0$$
$$= \frac{(s+1)^2 - s(s+1) - s}{s(s+1)^2)} = \frac{1}{s(s+1)^2}$$
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(s+1)^2} \qquad \mathcal{R}e\{s\} > -1$$

Using the output $y(t) = (2 - 3e^{-t} - e^{-3t})u(t)$

$$\begin{aligned}
Y(s) &= \frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+3} \qquad \mathcal{R}e\{s\} > 0 \\
&= \frac{2(s+1)(s+3-3s(s+3)-s(s+1)}{s(s+1)(s+3))} = \frac{6}{s(s+1)(s+3)} \\
X(s) &= \frac{Y(s)}{H(s)} = \frac{6(s+1)}{s(s+3)} = 6\left[\frac{A}{s} + \frac{B}{s+3}\right] \\
A &= \left.\frac{s+1}{s+3}\right|_{s=0} = \frac{1}{3} \\
B &= \left.\frac{s+1}{s}\right|_{s=-1} = \frac{-2}{-3} = \frac{2}{3} \\
X(s) &= \frac{2}{s} + \frac{4}{s+3} \qquad \mathcal{R}e\{s\} > 0 \\
x(t) &= 2u(t) + 4e^{-3t}u(t)
\end{aligned}$$

3. (OW 9.31)

Solution

(a) See Fig. 1 for the pole plot.

$$\frac{d^2y(t)}{dt^t} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$
$$[s^2 - s - 2]Y(s) = X(s)$$

$$\begin{aligned}
H(s) &= \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \\
A &= \left.\frac{1}{s+1}\right|_{s=2} = \frac{1}{3} \\
B &= \left.\frac{1}{s-2}\right|_{s=-1} = \frac{1}{-3} \\
H(s) &= \frac{1/3}{s-2} - \frac{1/3}{s+1}
\end{aligned}$$

(b) (i) stable $\Rightarrow -1 < \mathcal{R}e\{s\} < 2$

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)$$

(ii) causal $\Rightarrow \mathcal{R}e\{s\} > 2$

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t)$$

(iii) neither $\Rightarrow \mathcal{R}e\{s\} < -1$

$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t)$$

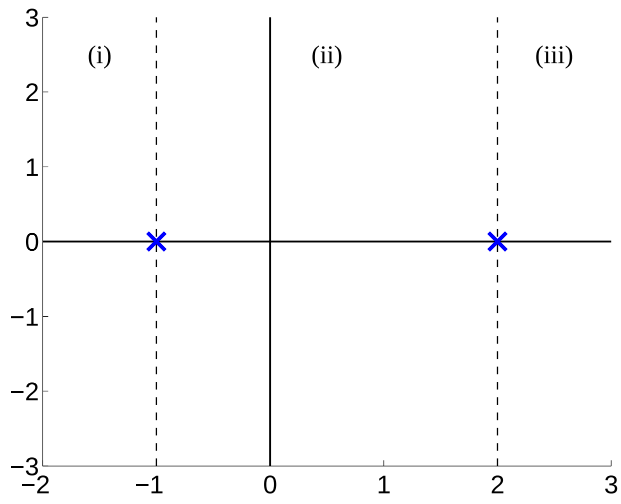

Figure 1: OW 9.31

4. (OW 9.33)

Solution

$$
\begin{aligned}
H(s) &= \frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1+j)(s+1-j)} \qquad \mathcal{R}e\{s\} > -1 \\
x(t) &= e^{-|t|} = e^t u(-t) + e^{-t}u(t) \\
X(s) &= \frac{-1}{s-1} + \frac{1}{s+1} = \frac{-2}{(s-1)(s+1)} \qquad -1 < \mathcal{R}e\{s\} < 1 \\
Y(s) &= X(s)H(s) = \left[\frac{-2}{(s-1)(s+1)}\right]\left[\frac{s+1}{s^2+2s+2}\right] = \frac{-2}{(s-1)(s^2+2s+2)} \\
&= \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+2} \\
&A = \left.\frac{-2}{s^2+2s+2}\right|_{s=1} = -\frac{2}{5} \\
&(Bs+C)(s-1) + A(s^2+2s+2) = -2 \\
&A + B = 0 \Rightarrow B = -A = \frac{2}{5} \\
&-C + 2A = -2 \Rightarrow C = 2A + 2 = \frac{6}{5} \\
Y(s) &= -\frac{2/5}{s-1} + \frac{2/5s+6/5}{s^2+2s+2} \qquad -1 < \mathcal{R}e\{s\} < 1 \\
&= -\frac{2/5}{s-1} + \frac{2}{5}\left[\frac{s+1}{(s+1)^2+1}\right] + \frac{4}{5}\left[\frac{1}{(s+1)^2+1}\right] \\
y(t) &= \frac{2}{5}e^t u(-t) + \frac{2}{5}e^{-t}\cos(t)u(t) + \frac{4}{5}e^{-t}\sin(t)u(t)
\end{aligned}
$$

5. (OW 9.40)

Solution

(a)

$$Y(s)[s^3+6s^2+11s+6] = X(s) = \frac{1}{s+4}$$

$$\begin{aligned}
H(s) &= \frac{Y(s)}{X(s)} = \frac{1}{s^3+6s^2+11s+6} = \frac{1}{(s+1)(s+2)(s+3)} \\
Y(s) &= X(s)H(s) = \frac{1}{(s+1)(s+2)(s+3)(s+4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4} \\
A &= \left.\frac{1}{(s+2)(s+3)(s+4)}\right|_{s=-1} = \frac{1}{6} \\
B &= \left.\frac{1}{(s+1)(s+3)(s+4)}\right|_{s=-2} = -\frac{1}{2} \\
C &= \left.\frac{1}{(s+1)(s+2)(s+4)}\right|_{s=-3} = \frac{1}{2} \\
D &= \left.\frac{1}{(s+1)(s+2)(s+3)}\right|_{s=-4} = -\frac{1}{6} \\
Y(s) &= \frac{1/6}{s+1} - \frac{1/2}{s+2} + \frac{1/2}{s+3} - \frac{1/6}{s+4}
\end{aligned}$$

Since a causal response is required for the zero-state response (ZSR)

$$y_{\text{zsr}}(t) = \left[\frac{1}{6}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-3t} - \frac{1}{6}e^{-4t}\right]u(t)$$

(b) The zero-input response (ZIR) uses the initial conditions and unilateral Laplace Transform with $X_u(s) = 0$.

$$\begin{aligned}
s^3Y_u(s) - s^2y(0^-) - sy'(0^-) - y''(0^-) + 6\left[s^2Y_u(s) - sy(0^-) - y'(0^-)\right] \\
+11\left[sY_u(s) - y(0^-)\right] + 6Y_u(s) = X_u(s)
\end{aligned}$$

$$\begin{aligned}
Y_u(s)\left[s^3+6s^2+11s+6\right] - \\
\left[s^2y(0^-) + s(y'(0^-)+6y(0^-)) + y''(0^-) + 6y'(0^-) - 11y(0^-)\right] &= 0 \\
Y_u(s)\left[s^3+6s^2+11s+6\right] - \left[s^2+5s+6\right] &= 0 \\
Y_u(s)\left[s^3+6s^2+11s+6\right] &= s^2+5s+6
\end{aligned}$$

$$\begin{aligned}
Y_u(s) &= \frac{s^2+5s+6}{s^3+6s^2+11s+6} = \frac{(s+3)(s+2)}{(s+1)(s+2)(s+3)} = \frac{1}{s+1} \\
y_{zir}(t) &= e^{-t}u(t)
\end{aligned}$$

(c) The total response is the sum of the ZSR and ZIR (for input $x(t) = e^{-4t}u(t)$)

$$y(t) = y_{zsr}(t) + y_{zir}(t) = \left[\frac{7}{6}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-3t} - \frac{1}{6}e^{-4t}\right]u(t)$$

6. (OW 9.45)

Solution

(a)

$$X(s) = \frac{s+2}{s-2} \qquad \mathcal{R}e\{s\} < 2$$
$$y(t) = -\frac{2}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(t)$$
$$Y(s) = \frac{2/3}{s-2} + \frac{1/3}{s+1} = \frac{s}{(s+1)(s-2)} \qquad -1 < \mathcal{R}e\{s\} < 2$$
$$H(s) = \frac{Y(s)}{X(s)} = \frac{s}{(s+1)(s+2)} \qquad \mathcal{R}e\{s\} > -1$$

(b)

$$H(s) = \frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$
$$= A = \left.\frac{s}{s+2}\right|_{s=-1} = -1$$
$$= B = \left.\frac{s}{s+1}\right|_{s=-2} = 2$$
$$H(s) = \frac{-1}{s+1} + \frac{2}{s+2}$$
$$h(t) = -e^{-t}u(t) + 2e^{-2t}u(t)$$

(c) Remember e^{st} is an eigen-signal for an LTI system and use $H(s)$ from above to solve.

$$x(t) = e^{3t} \longleftrightarrow y(t) = \left. H(s)\right|_{s=3} e^{3t} = \frac{3}{20}e^{3t}$$

7. (OW 9.48)

Solution

(a)

$$\delta(t) = h(t) * h_1(t) \overset{\mathcal{L}}{\longleftrightarrow} 1 = H(s)H_1(s)$$
$$H_1(s) = \frac{1}{H(s)}$$

(b) See the pole-zero plot in Fig. 2.

$$H_1(s) = \frac{1}{H(s)} = \frac{s+1}{s-1/2}$$

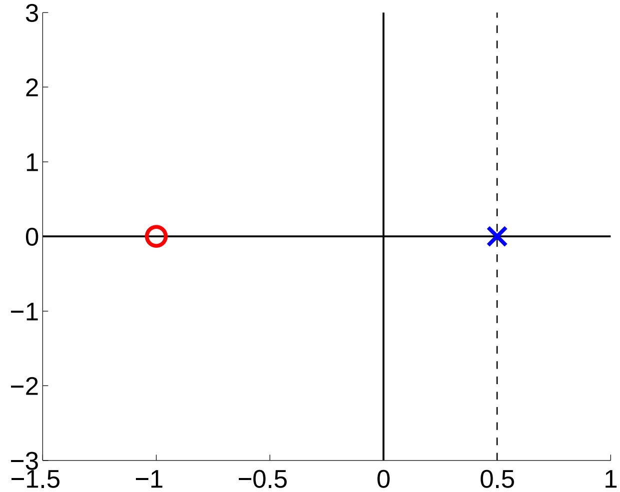

Figure 2: OW 9.48