Homework #5
Due Su. 3/24

Note: The **Basic Problems with Answers** will be worth half as much as the other questions. You must show all your work to receive credit.

1. (OW 9.7) **Basic Problem**

   Solution

   The different signals with Laplace Transform

$$X(s) = \frac{(s-1)}{(s+2)(s+3)(s^2+s+1)}$$

are found by examining the pole-zero plot. The 4 poles of of $X(s)$ are located at

$$s_0 = -2 \qquad s_1 = -3 \qquad s_2 = \frac{1}{2}(-1+j\sqrt{3}) \qquad s_3 = \frac{1}{2}(-1-j\sqrt{3})$$

where s_2 and s_3 are found using the Quadratic Formula. The four poles results in four regions of convergence as shown in Fig. 3.7 (notice the complex conjugate pair only adds one constraint on the $\mathcal{R}e\{s\}$)

(I) $\mathcal{R}e\{s\} < -3$
(II) $-3 < \mathcal{R}e\{s\} < -2$
(III) $-2 < \mathcal{R}e\{s\} < -\frac{1}{2}$
(IV) $\mathcal{R}e\{s\} > -\frac{1}{2}$

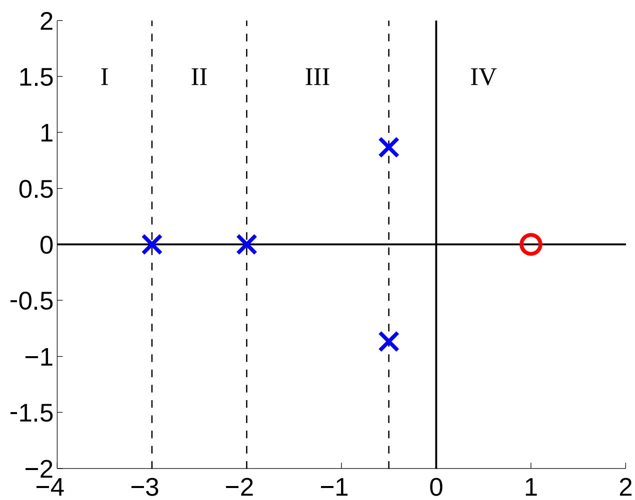

Figure 3.7: OW 9.7

2. (OW 9.9) **Basic Problem**

   Solution

$$X(s) = \frac{2(s+2)}{s^2+7s+12} = \frac{2(s+2)}{(s+4)(s+3)} = \frac{A}{s+4} + \frac{B}{s+3}$$

$$A = \left.\frac{2(s+2)}{s+3}\right|_{s=-4} = \frac{2(-2)}{-1} = 4$$

$$B = \left.\frac{2(s+2)}{s+4}\right|_{s=-3} = \frac{2(-1)}{1} = -2$$

$$X(s) = \frac{4}{s+4} - \frac{2}{s+3}$$
$$x(t) = 4e^{-4t}u(t) - 2e^{-3t}u(t)$$

3. (OW 9.21 (a),(b),(c),(d),(g))

   Solution

   (a) $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$

   $$X(s) = \frac{1}{s+2} + \frac{1}{s+3} = \frac{s+3+2+2}{(s+2)(s+3)} = \frac{2s+5}{(s+2)(s+3)} \qquad \mathcal{R}e\{s\} > -2$$

   (b) $x(t) = e^{-4t}u(t) + e^{-5t}\sin(5t)u(t)$

   $$\begin{aligned} X(s) &= \underbrace{\frac{1}{s+4}}_{\mathcal{R}e\{s\}>-4} + \underbrace{\frac{5}{(s+5)^2+5^2}}_{\mathcal{R}e\{s\}>-5} \qquad \mathcal{R}e\{s\} > -4 \\ &= \frac{1}{s+4} + \frac{5}{s^2+10s+50} \\ &= \frac{s^2+15s+70}{(s+4)(s^2+10s+50)} \\ &= \frac{(s+0.5(15+\sqrt{55}))(s+0.5(15-\sqrt{55}))}{(s+4)(s+5-5j)(s+5+5j)} \end{aligned}$$

   (c) $x(t) = e^{2t}u(-t) + e^{3t}u(-t)$

   $$X(s) = -\underbrace{\frac{1}{s-2}}_{\mathcal{R}e\{s\}<2} - \underbrace{\frac{1}{s-3}}_{\mathcal{R}e\{s\}<3} = -\frac{2s-5}{(s-2)(s-3)} \qquad \mathcal{R}e\{s\} < 2$$

   (d) $x(t) = te^{-2|t|} = te^{2t}u(-t) + te^{-2t}u(t)$

   $$\begin{aligned} X(s) &= -\underbrace{\frac{1}{(s-2)^2}}_{\mathcal{R}e\{s\}<2} - \underbrace{\frac{1}{(s+2)^2}}_{\mathcal{R}e\{s\}>-2} = \frac{-(s+2)^2+(s-2)^2}{(s-2)^2(s+2)^2} \qquad -2 < \mathcal{R}e\{s\} < 2 \\ &= \frac{-8s}{(s-2)^2(s+2)^2} \end{aligned}$$

   (g) $x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{else} \end{cases} = u(t) - u(t-1)$

   $$X(s) = \frac{1}{s} - e^{-s}\frac{1}{s} = \frac{1}{s}(1-e^{-s}) \qquad \forall s$$

   The ROC is the entire s-plane because the signal is of finite duration. This can be confusing due to the s in the denominator. However, you can see that the numerator has a zero at $z = 0$ so there is pole-zero cancellation.

4. (OW 9.22 (b),(c),(d),(f),(g))

   Solution

(b)

$$\frac{s}{s^2+9}, \mathcal{R}e\{s\} < 0 \overset{\mathcal{L}}{\longleftrightarrow} x(t) = -\cos(3t)u(-t)$$

(c)

$$\frac{s+1}{(s+1)^2+9}, \mathcal{R}e\{s\} < -1 \overset{\mathcal{L}}{\longleftrightarrow} x(t) = -e^{-t}\cos(3t)u(-t)$$

(d) With ROC: $-4 \leq \mathcal{R}e\{s\} \leq -3, x(t)$ is two-sided

$$\begin{aligned}
X(s) &= \frac{s+2}{s^2+7s+12} = \frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4} \\
A &= \left.\frac{s+2}{s+4}\right|_{s=-3} = \frac{-1}{1} = -1 \\
B &= \left.\frac{s+2}{s+3}\right|_{s=-4} = \frac{-2}{-1} = 2 \\
X(s) &= \frac{-1}{s+3} + \frac{2}{s+4} \\
x(t) &= e^{-3t}u(-t) + 2e^{-4t}u(t)
\end{aligned}$$

(f) The trick here is to recognize that the complex conjugate pairs give rise to (co)sine terms.

$$\begin{aligned}
X(s) &= \frac{(s+1)^2}{s^2-s+1} = 1 + \frac{3s}{s^2-s+1} = 1 + \frac{3s}{(s-\frac{1}{2}(1+j\sqrt{3}))(s-\frac{1}{2}(1-j\sqrt{3}))} \\
&= 1 + \frac{3s}{(s-1/2)^2+(\sqrt{3}/2)^2} \\
&= 1 + \frac{3(s-1/2)}{(s-1/2)^2+(\sqrt{3}/2)^2} + \frac{3/2}{(s-1/2)^2+(\sqrt{3}/2)^2} \\
x(t) &= \delta(t) + e^{-t/2}\cos(\sqrt{3}t/2)u(t) + \sqrt{3}e^{-t/2}\sin(\sqrt{3}t/2)u(t)
\end{aligned}$$

(g)

$$\begin{aligned}
X(s) &= \frac{s^2-s+1}{(s+1)^2} = 1 - \frac{3s}{(s+1)^2} = 1 + \frac{A}{(s+1)^2} + \frac{B}{s+1} \\
A &= \left.-3s\right|_{s=-1} = 3 \\
B &= \left.\frac{d}{ds} - 3s\right|_{s=-1} = -3 \\
X(s) &= 1 + \frac{3}{(s+1)^2} - \frac{3}{s+1} \\
x(t) &= \delta(t) + 3te^{-t}u(t) - 3e^{-t}u(t)
\end{aligned}$$

5. (OW 9.26)

Solution

Given

$$y(t) = x_1(t-2) * x_2(-t+3),$$

the Laplace Transform can be found by examining each term individually using the transform properties.

$$\begin{aligned}
x_1(t) = e^{-2t}u(t) &\overset{\mathcal{L}}{\longleftrightarrow} X_1(s) = \frac{1}{s+2}, && \mathcal{R}e\{s\} > -2 \\
x_1(t-3) &\overset{\mathcal{L}}{\longleftrightarrow} e^{-2s}X_1(s) = \frac{e^{-2s}}{s+2}, && \mathcal{R}e\{s\} > -2 \\
x_2(t) = e^{-3t}u(t) &\overset{\mathcal{L}}{\longleftrightarrow} X_1(s) = \frac{1}{s+3}, && \mathcal{R}e\{s\} > -3 \\
g(t) = x_2(t+3) &\overset{\mathcal{L}}{\longleftrightarrow} G(s) = e^{3s}X_2(s), && \mathcal{R}e\{s\} > -3 \\
x_2(-t+3) = g(-t) &\overset{\mathcal{L}}{\longleftrightarrow} G(-s) = e^{-3s}X_2(-s) = \frac{e^{-3s}}{-s+3}, && \mathcal{R}e\{s\} < 3
\end{aligned}$$

$$Y(s) = \left[\frac{e^{-2s}}{s+2}\right]\left[\frac{e^{-3s}}{3-s}\right], \quad -2 < \mathcal{R}e\{s\} < 3$$

6. (OW 9.27)

   Solution

   From clues 1 and 2, the LT has the form

   $$X(s) = \frac{A}{(s+s_0)(s+s_1)}$$

   because there are two poles and no zeros (A is a scale factor). With clue 3 and because $x(t)$ is real, the poles must be complex conjugates and

   $$X(s) = \frac{A}{(s+s_0)(s+s_0^*)} = \frac{A}{(s+1+j)(s+1-j)}$$

   Using clue 5 with $X(s)|_{s=0} = 8$, the final algebraic form is

   $$X(s) = \frac{16}{(s+1+j)(s+1-j)}$$

   The ROC can be found by using clue 4 and noting that

   $$y(t) = e^{2t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = X(s-2), \quad \text{ROC}_x + 2$$

   must not include the $j\omega$-axis. This implies the ROC of $X(s)$ should be right-sided

   $$X(s) = \frac{16}{(s+1+j)(s+1-j)} \quad \mathcal{R}e\{s\} > -1$$

7. (OW 9.28)

   Also, give the algebraic form of $H(s)$.

   Solution

   There are four possible ROC associated with the system function as noted in Fig. 3.7

   $$H(s) = \frac{s-2}{(s-1)(s+1)(s+2)}.$$

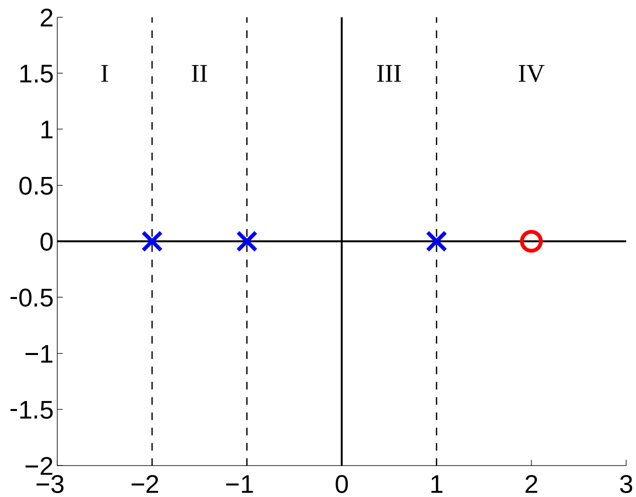

Figure 3.7: OW 9.28

(a) The four ROC are

(I) $\mathcal{R}e\{s\} < -2$

(II) $-2 < \mathcal{R}e\{s\} < -1$

(III) $-1 < \mathcal{R}e\{s\} < 1$

(IV) $\mathcal{R}e\{s\} > 1$

(b) For a rational $H(s)$, stability implies the $j\omega$-axis is in the ROC and causality implies a right-sided ROC to the right of rightmost pole ($\mathcal{R}e\{s\}= 1$).

(I) not stable, not causal

(II) not stable, not causal

(III) stable, not causal

(IV) not stable, causal