Homework #1
Due Su. 1/28

1. (OW 1.21)

   Solution

   See Fig. 1 for plots.

   (a) Shift right by 1.

   (b) Shift left by 2 and flip.

   (c) Shift left 1 and squeeze by factor of 2.

   (d) Shift left by 4, flip, and stretch by factor of 2.

   (e) $x(t)u(t) + x(-t)u(t)$

   (f) $x(t)\delta(t + \frac{3}{2}) - x(t)\delta(t - \frac{3}{2}) = x(\frac{-3}{2})\delta(t + \frac{3}{2}) - x(\frac{3}{2})\delta(t - \frac{3}{2})) = -0.5\delta(t + \frac{3}{2}) - 0.5\delta(t - \frac{3}{2}))$
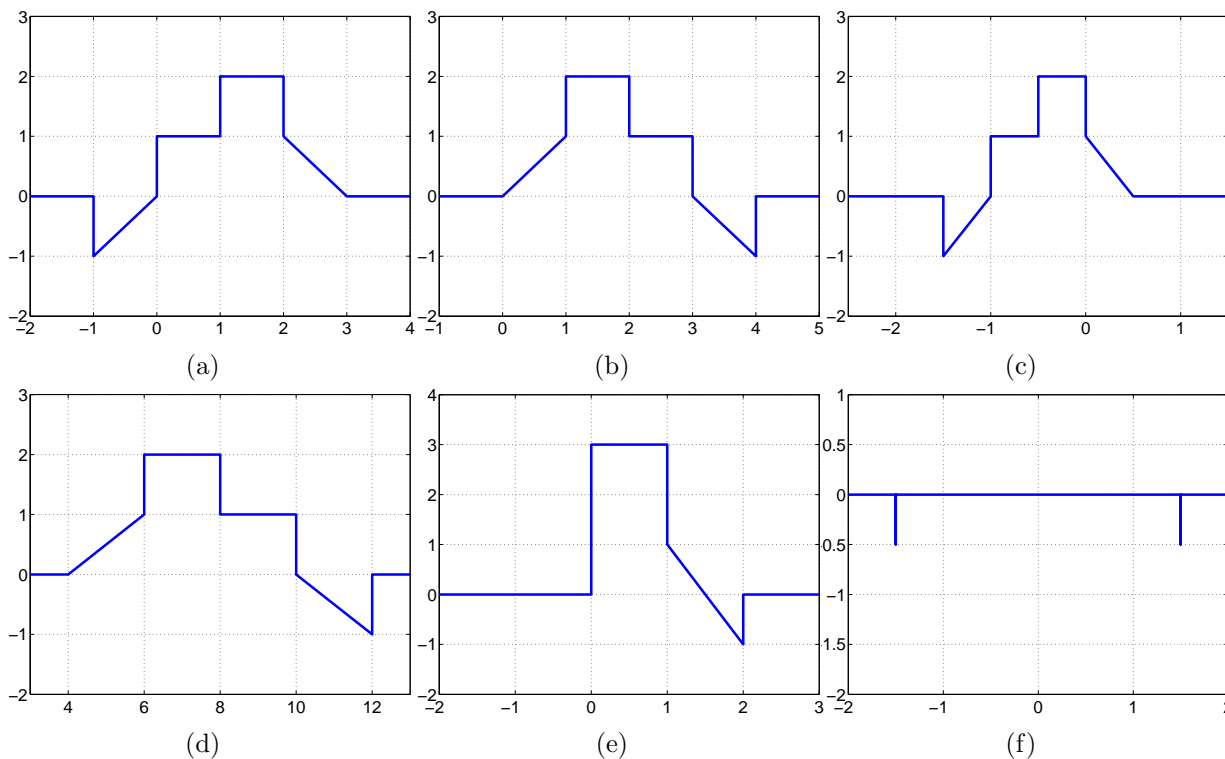

Figure 1: OW 1.21 Solutions

2. (OW 1.22 (a)-(f))

   Solution

   See Fig. 2 for plots.

   (a) Shift right by 4.

   (b) Shift left by 3 and flip.

   (c) Decimate (squeeze) by 3. Only keep integer indicies.

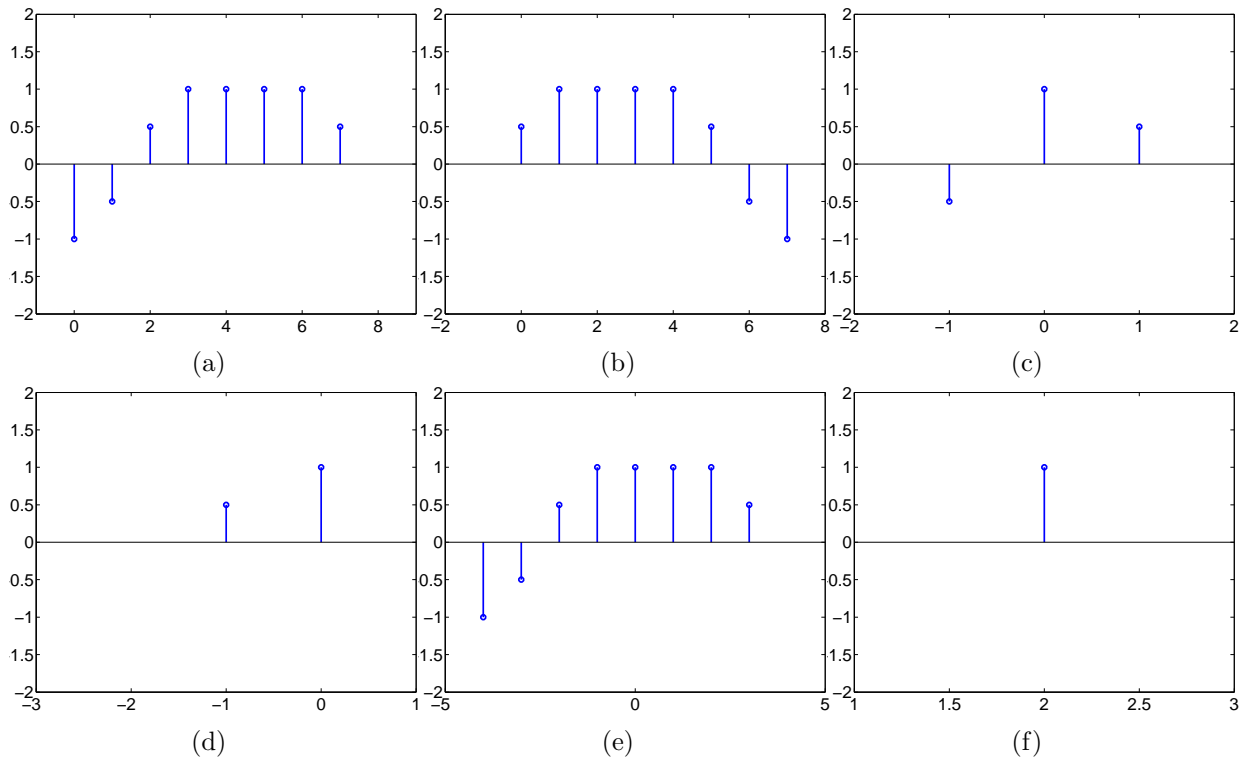   (d) Shift left by 1 and decimate by 3.

Figure 2: OW 1.22 Solutions

(e) $x[n]u[3-n] = x[n]$ since $u[3-n] = 1$ for $n \leq 3$.

(f) $x[n-2]\delta[n-2] = x[2-2]\delta[n-2] = x[0]\delta[n-2] = \delta[n-2]$

3. (OW 1.31)

   Solution

   See Fig. 3 for plots.

   (a) $x_2(t) = x_1(t) - x_1(t-2) \longrightarrow y_2(t) = y_1(t) - y_1(t-2)$

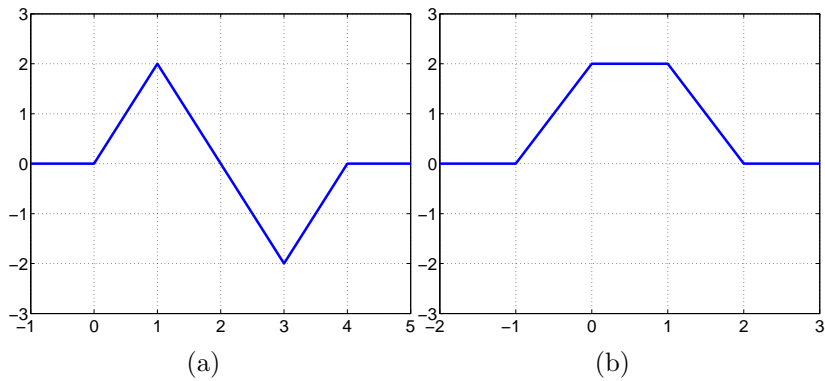   (b) $x_3(t) = x_1(t+1) + x_1(t) \longrightarrow y_3(t) = y_1(t+1) + y_1(t)$

Figure 3: OW 1.31 Solutions

4. (OW 1.49 (a)-(f), (i), (k), (l))

Solution

Convert the Cartesian coordinates ($z = x + jy$) into polar using the formulas

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The points are plotted in the complex plane in Fig. 4.

(a) $1 + j\sqrt{3} = \sqrt{1 + (\sqrt{3})^2} e^{j\tan^{-1}(\frac{\sqrt{3}}{1})} = 2e^{j\pi/3}$

(b) $-5 = 5e^{j\pi}$

(c) $-5 - 5j = 5\sqrt{2}e^{j5\pi/4}$

(d) $3 + 4j = 5e^{j\tan^{-1}(\frac{4}{3})}$

(e) $(1 - j\sqrt{3})^3 = (1 - j2\sqrt{3} - 3)(1 - j\sqrt{3}) = -2(1 + j\sqrt{3})(1 - j\sqrt{3}) = -2(1 + 3) = -8 = 8e^{j\pi}$

(f) $(1 - j)^5 = (1 - j)^2(1 - j)^2(1 - j) = (-2j)^2(1 - j) = 4j^2(1 - j) = -4(1 - j) = -4 + 4j = 4\sqrt{2}e^{j3\pi/4}$

(i) $\dfrac{1 + j\sqrt{3}}{\sqrt{3} + j} = \dfrac{\sqrt{3} + 3j - j + \sqrt{3}}{4} = \dfrac{2\sqrt{3} + 2j}{4} = \dfrac{1}{2}(\sqrt{3} + j) = e^{j\pi/6}$

(k) Using results from (i)
$(\sqrt{3} + j)2\sqrt{2}e^{-j\pi/4} = (2e^{j\pi/6})2\sqrt{2}e^{-j\pi/4} = 4\sqrt{2}e^{-j\pi/12}$

(l) Using results from (i)
$\dfrac{e^{j\pi/3} - 1}{1 + j\sqrt{3}} = \dfrac{(2 + j2\sqrt{3}) - 1}{1 + j\sqrt{3}} = \dfrac{1}{2}\left(\dfrac{-1 + j\sqrt{3}}{1 + j\sqrt{3}}\right) = \dfrac{1}{4}(1 + j\sqrt{3}) = \dfrac{1}{2}e^{j\pi/3}$
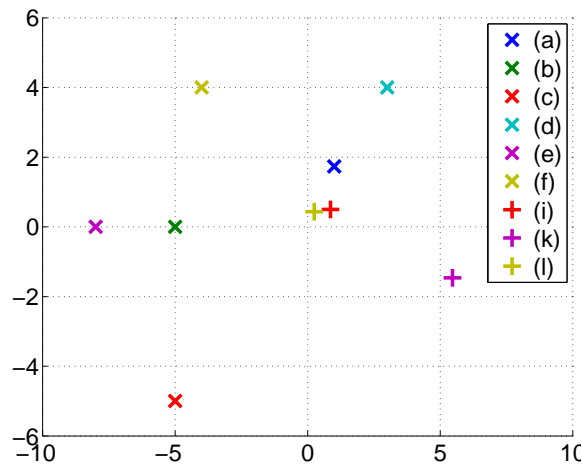

Figure 4: OW 1.49 Solutions

5. (OW 1.51)

   Solution

   (a)

   $$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$
   $$= \frac{1}{2}(\cos\theta + j\sin\theta + \cos\theta - j\sin\theta)$$
   $$= \frac{1}{2}(2\cos\theta)$$
   $$= \cos\theta$$

   (b)

   $$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$
   $$= \frac{1}{2j}(\cos\theta + j\sin\theta - (\cos\theta - j\sin\theta))$$
   $$= \frac{1}{2j}(2j\sin\theta)$$
   $$= \sin\theta$$

   (c)

   $$\cos^2\theta = \left(\frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)\right)^2$$
   $$= \frac{1}{4}\left(e^{j2\theta} + e^{j\theta}e^{-j\theta} + e^{-j\theta}e^{j\theta} + e^{-j2\theta}\right)$$
   $$= \frac{1}{4}\left(e^{j2\theta} + e^{-j2\theta}\right)\frac{1}{4}\left(e^{j0} + e^{j0}\right)$$
   $$= \frac{1}{2}\cdot\frac{1}{2}\left(e^{j2\theta} + e^{-j2\theta}\right) + \frac{1}{4}(2)$$
   $$= \frac{1}{2}\cos 2\theta + \frac{1}{2}$$
   $$= \frac{1}{2}(1 + \cos 2\theta)$$

   (d)

   $$(\sin\theta)(\sin\phi) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)\frac{1}{2j}\left(e^{j\phi} - e^{-j\phi}\right)$$
   $$= \frac{1}{4j^2}\left(e^{j(\theta+\phi)} - e^{j(\theta-\phi)} - e^{-j(\theta-\phi)} + e^{-j(\theta+\phi)}\right)$$
   $$= \frac{1}{4}\left(e^{j(\theta-\phi)} + e^{-j(\theta-\phi)}\right) - \frac{1}{4}\left(e^{j(\theta+\phi)} + e^{-j(\theta+\phi)}\right)$$
   $$= \frac{1}{2}\cos(\theta - \phi) - \frac{1}{2}\cos(\theta + \phi)$$

(e) This can be best addressed working from the right-hand side of the equation.

$$\sin\theta\cos\phi + \cos\theta\sin\phi = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)\frac{1}{2}\left(e^{j\phi} + e^{-j\phi}\right) + \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)\frac{1}{2j}\left(e^{j\phi} - e^{-j\phi}\right)$$

$$= \frac{1}{4j}\left(e^{j(\theta+\phi)} + e^{j(\theta-\phi)} - e^{-j(\theta-\phi)} - e^{-j(\theta+\phi)}\right)$$

$$+ \frac{1}{4j}\left(e^{j(\theta+\phi)} - e^{j(\theta-\phi)} + e^{-j(\theta-\phi)} - e^{-j(\theta+\phi)}\right)$$

$$= \frac{1}{4j}\left(2e^{j(\theta+\phi)} - 2e^{-j(\theta+\phi)}\right)$$

$$= \frac{1}{2j}\left(e^{j(\theta+\phi)} - e^{-j(\theta+\phi)}\right)$$

$$= \sin(\theta + \phi)$$

6. (OW 1.55 (a)-(e))

   Solution

   Use the results from Problem 1.54 for summation formulas.

   (a)
   $$\sum_{n=0}^{9} e^{j\pi n/2} = \frac{1 - e^{j\pi(10)/2}}{1 - e^{j\pi/2}} = \frac{1 - e^{j5\pi}}{1 - j} = \frac{1 - (-1)}{1 - j} = \frac{2}{1 - j} = 1 + j$$

   (b) Using results from (a)
   $$\sum_{n=-2}^{7} e^{j\pi n/2} = \sum_{l=0}^{9} e^{j\pi(l+2)/2} = e^{-j\pi}\sum_{l=0}^{9} e^{j\pi l/2} = -1 \times \sum_{l=0}^{9} e^{j\pi l/2} = -1(1 + j) = -1 - j$$

   (c)
   $$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{j\pi n/2} = \frac{1}{1 - \frac{1}{2}e^{j\pi/2}} = \frac{1}{1 - \frac{1}{2}j} = \frac{1}{5}(4 + 2j)$$

   (d) Using results from (c) and Problem 8 below
   $$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n e^{j\pi n/2} = \frac{(\frac{1}{2}e^{j\pi/2})}{1 - \frac{1}{2}j} = \frac{j^2}{4}\{\frac{1}{5}(4 + 2j)\} = -\frac{1}{4}\{\frac{1}{5}(4 + 2j)\} = -\frac{1}{5} - \frac{j}{10}$$

   (e) Using results from (a)
   $$\sum_{n=0}^{9} \cos(\frac{\pi}{2}n) = \sum_{n=0}^{9} \frac{1}{2}e^{j\pi n/2} + \frac{1}{2}e^{-j\pi n/2} = \frac{1}{2}(1 + j) + \frac{1}{2}(1 - j) = 1$$

7. (OW 1.56 (b)-(f))

   Solution

   (b)
   $$\int_0^6 e^{j\pi t/2}dt = \left[\frac{1}{j\pi/2}e^{j\pi t/2}\right]_0^6 = \frac{1}{j\pi/2}\left[e^{j3\pi} - 1\right] = \frac{1}{j\pi/2}[-2] = -\frac{4}{j\pi} = \frac{4j}{\pi}$$

   (c)
   $$\int_2^8 e^{j\pi t/2}dt = \left[\frac{1}{j\pi/2}e^{j\pi t/2}\right]_2^8 = \frac{1}{j\pi/2}\left[e^{j4\pi} - e^{j\pi}\right] = \frac{1}{j\pi/2}[1 - (-1)] = \frac{4}{j\pi} = -\frac{4j}{\pi}$$

   (d)
   $$\int_0^{\infty} e^{-(1+j)t}dt = \left[-\frac{1}{1+j}e^{-(1+j)t}\right]_0^{\infty} = -\frac{1}{1+j}[0 - 1] = \frac{1}{1+j} = \frac{1-j}{2}$$

(e) Using result from part (d)
$$\int_0^\infty e^t \cos(t)dt = \frac{1}{2}\int_0^\infty \left[e^{-(1+j)t} + e^{-(1-j)t}\right]dt = \frac{1}{2}\left[\frac{1+j}{2} + \frac{1-j}{2}\right] = \frac{1}{2}$$

(f) Using a similar approach as used in part (e)

$$\begin{aligned}
\int_0^\infty e^t \sin(t)dt &= \frac{1}{2j}\int_0^\infty e^{-(2-3j)t} + e^{-(2+3j)t}dt \\
&= \frac{1}{2j}\left\{\left[-\frac{1}{2-3j}e^{-(2-3j)t}\right]_0^\infty + \left[\frac{1}{2+3j}e^{-(2+3j)t}\right]_0^\infty\right\} \\
&= \frac{1}{2j}\left[\frac{1}{2-3j} - \frac{1}{2+3j}\right] \\
&= \frac{3}{13}
\end{aligned}$$

8. Using expressions in OW 1.54 and for any $0 < N_1, N_2 < \infty$,

   (a) For $a \neq 1$, find a closed form expression for
   $$\sum_{n=N_1}^{N_2} a^n$$

   (b) For $|a| < 1$, find a closed form expression for
   $$\sum_{n=N_1}^{\infty} a^n.$$

   Solution

   (a) Let $k = n - N_1$,
   $$\begin{aligned}
   \sum_{n=N_1}^{N_2} a^n &= \sum_{k=0}^{N_2-N_1} a^{k+N_1} \\
   &= \sum_{k=0}^{N_2-N_1} a^k a^{N_1} = a^{N_1}\sum_{k=0}^{N_2-N_1} a^k \\
   &= a^{N_1}\frac{1 - a^{N_2-N_1+1}}{1-a} \\
   &= \frac{a^{N_1} - a^{N_2+1}}{1-a}
   \end{aligned}$$

   (b) Let $k = n - N_1$,
   $$\begin{aligned}
   \sum_{n=N_1}^{\infty} a^n &= \sum_{k=0}^{\infty} a^{k+N_1} \\
   &= a^{N_1}\sum_{k=0}^{\infty} a^k \\
   &= a^{N_1}\frac{1}{1-a} \\
   &= \frac{a^{N_1}}{1-a}
   \end{aligned}$$