EE292: Fundamentals of ECE

Fall 2012
TTh 10:00-11:15 SEB 1242

Lecture 18
121025

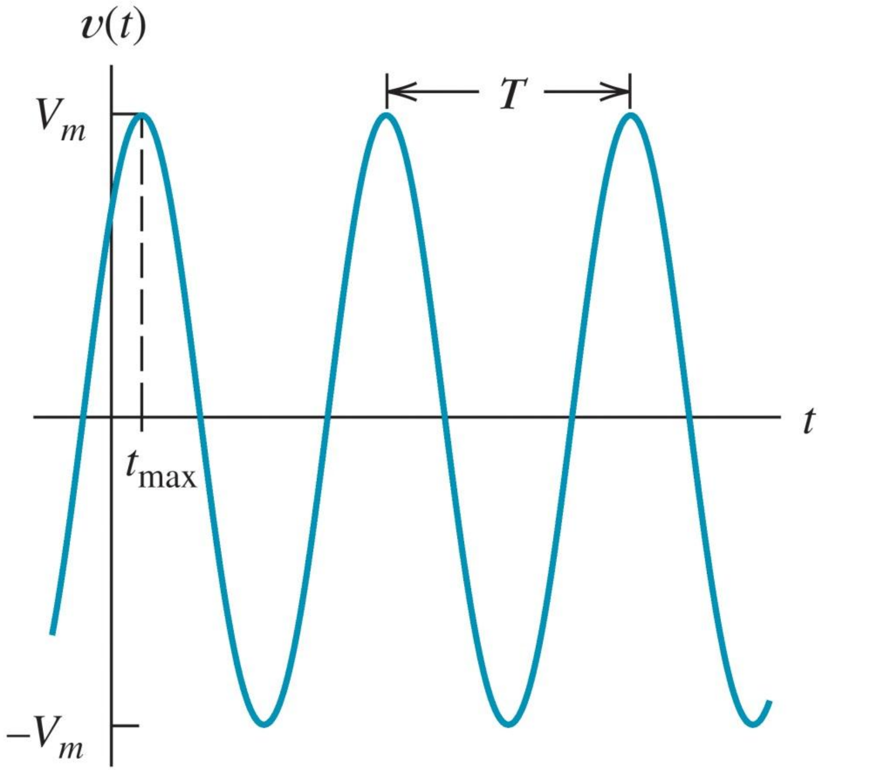
http://www.ee.unlv.edu/~b1morris/ee292/

Outline

- Review
  - RMS Values
  - Complex Numbers
- Phasors
- Complex Impedance
- Circuit Analysis with Complex Impedance

Sinusoidal Currents and Voltages

- Sinusoidal voltage
  - $v(t) = V_m \cos(\omega_0 t + \theta)$
  - V_m - peak value of voltage
  - ω_0 - angular frequency in radians/sec
  - θ – phase angle in radians
- This is a periodic signal described by
  - T – the period in seconds
    - $\omega_0 = \frac{2\pi}{T}$
  - f – the frequency in Hz = 1/sec
    - $f = \frac{1}{T}$
    - $\omega_0 = 2\pi f$



- Note: Assuming that θ is in degrees, we have
  - tmax = –θ/360 × T.

Sinusoidal Currents and Voltages

- For consistency/uniformity always express a sinusoid as a cosine
- Convert between sine and cosine
- In Degrees
  - $\sin(x) = \cos(x - 90^{\circ})$
- In Radians
  - $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$

- $v(t) = 10\sin(200t + 30^{\circ})$
- $v(t) = 10\cos(200t + 30^{\circ} - 90^{\circ})$
- $v(t) = 10\cos(200t - 60^{\circ})$

Root-Mean-Square Values

- Apply a sinusoidal source to a resistance
- Power absorbed
  - $p(t) = \frac{v^2(t)}{R}$
- Energy in a single period
  - $E_T = \int_0^T p(t)dt$
- Average power (power absorbed in a single period)
  - $P_{avg} = \frac{E_T}{T} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T \frac{v^2(t)}{R}dt$
  - $P_{avg} = \frac{\left[\sqrt{\frac{1}{T}\int_0^T v^2(t)dt}\right]^2}{R}$

Root-Mean-Square Values

- $P_{avg} = \frac{\left[\sqrt{\frac{1}{T}\int_0^T v^2(t)dt}\right]^2}{R}$
- Define rms voltage
  - $V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$

$$P_{avg} = \frac{V_{rms}^2}{R}$$

- Similarly define rms current
  - $I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt}$

$$P_{avg} = I_{rms}^2 R$$

RMS Value of a Sinusoid

- Given a sinusoidal source
  - $v(t) = V_m \cos(\omega_0 t + \theta)$
- $V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega_0 t + \theta)dt}$$

$$\text{using } \cos^2(x) = 1/2 + 1/2\cos(2x)$$

$$= \sqrt{\frac{V_m^2}{2T}\int_0^T [1 + \cos(2\omega_0 t + 2\theta)]\, dt}$$

$$\vdots$$

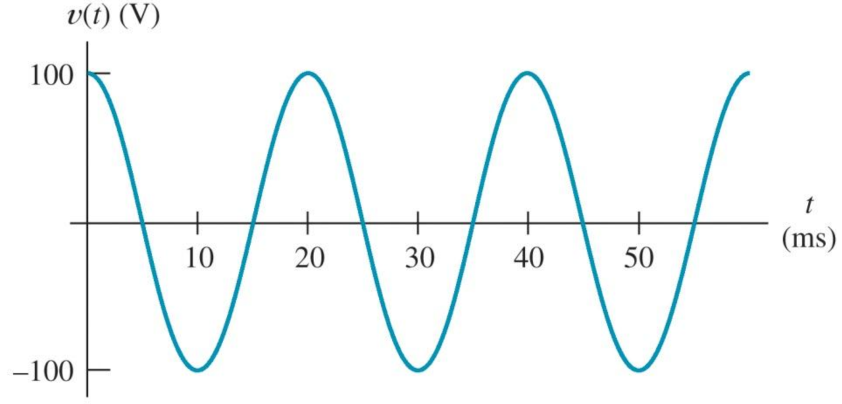
$$= \frac{V_m}{\sqrt{2}}$$

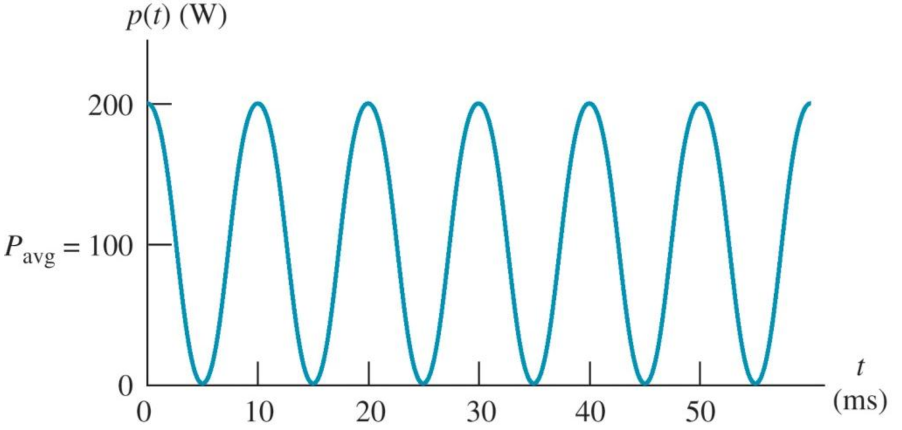
- The rms value is an "effective" value for the signal
  - E.g. in homes we have 60Hz 115 V rms power
  - $V_m = \sqrt{2} \cdot V_{rms} = 163\ V$

Example 5.1

- Voltage
  - $v(t) = 100\cos(100\pi t)$ V
- Applied to a 50Ω resistance

- Find the rms voltage and average power and plot power
- $\omega_0 = 100\pi$
- $f = \frac{\omega}{2\pi} = 50$ Hz
- $T = \frac{1}{f} = \frac{1}{50} = 20$msec
- $p(t) = \frac{v^2(t)}{R} = \frac{1}{50}100^2\cos^2(100\pi t) = 200\cos^2(100\pi t)$ W
- $V_{rms} = \frac{V_m}{\sqrt{2}} = \frac{100}{\sqrt{2}} = 70.71\ V$
- $P_{avg} = \quad \frac{V_{rms}^2}{R} \quad \frac{70.71^2}{50} = 100$ W

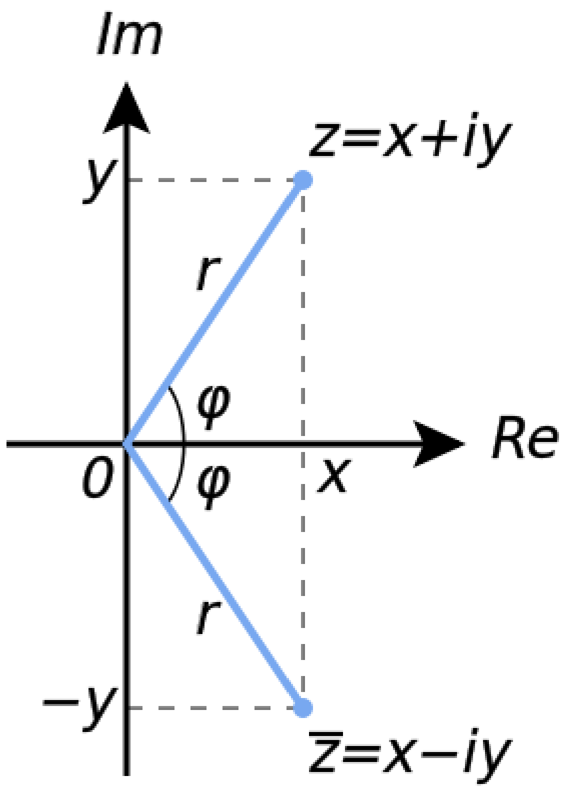




Complex Numbers in Rectangular Form

- A complex number in rectangular form
  $$z = x + jy$$
  - x – real part
  - y – imaginary part

- A vector in the complex plane with
  - x – horizontal coordinate
  - y – vertical coordinate



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Complex Arithmetic in Rectangular Form

- $z_1 = 5 + j5, \quad z_2 = 3 - j4$
- Addition
  - Add the real and complex parts separately
- $z_1 + z_2 = (5 + j5) + (3 - j4)$
- $z_1 + z_2 = (8 + j)$
- $z_1 - z_2 = (5 + j5) - (3 - j4)$
- $z_1 - z_2 = (2 + j9)$
- Multiplication
  - Use $j^2 = -1$
- $z_1 z_2 = (5 + j5)(3 - j4)$
- $= 15 - j20 + j15 - j^2 20$
- $= 15 - j5 - (-1)20$
- $= 35 - j5$
- Division
  - Multiply num/den by complex conjugate term to remove imaginary term in denominator
- $\frac{z_1}{z_2} = \frac{(5+j5)}{(3-j4)}$

Complex Arithmetic in Rectangular Form

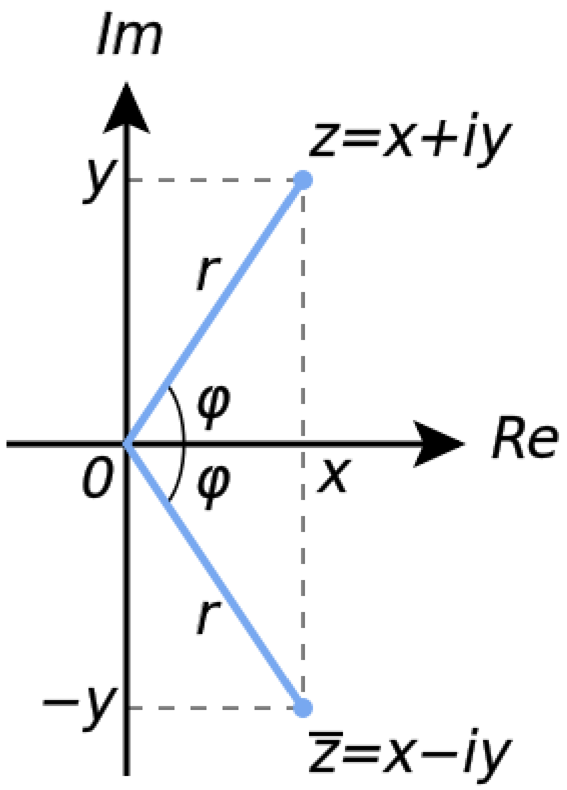
- Division
  - Multiply num/den by complex conjugate term to remove imaginary term in denominator

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{5+j5}{3-j4} \cdot \frac{z_2^*}{z_2^*} \\
&= \frac{5+j5}{3-j4} \cdot \frac{3+j4}{3+j4} \\
&= \frac{15+j20+j15+j^2(20)}{9-j12+j12-j^2(16)} \\
&= \frac{-5+j35}{25} \\
&= -0.2+j1.4
\end{aligned}
$$

Complex Numbers in Polar Form

- Number represented by magnitude and phase
- Magnitude – the length of the complex vector
- Phase – the angle between the real axis and the vector

- Polar notation
- $z = re^{j\theta}$
- Phasor notation
- $z = r\angle\theta$



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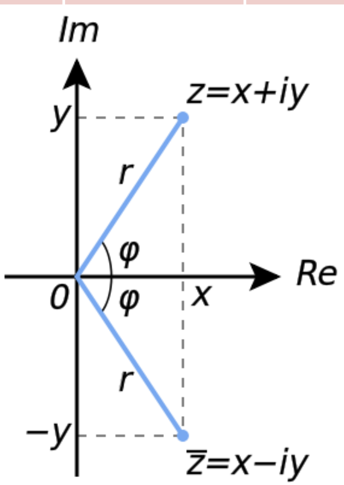
Conversion Between Forms

- Rectangular to polar form
- $r^2 = x^2 + y^2$
- $\tan\theta = \frac{y}{x}$

- Polar to rectangular form
- $x = r\cos\theta$
- $y = r\sin\theta$

- Convert to polar form
- $z = 4 - j4$
- $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$
- $\theta = \arctan\left(\frac{y}{x}\right) = \arctan(-1) = -\frac{\pi}{4}$
- $z = 4\sqrt{2}e^{-j\pi/4}$

x **(degrees)**	x **(radians)**	$\sin(x)$	$\cos(x)$	$\tan(x)$
0	0	0	1	0
15	$\frac{\pi}{12}$	$\frac{-1+\sqrt{3}}{2\sqrt{2}}$	$\frac{1+\sqrt{3}}{2\sqrt{2}}$	$2-\sqrt{3}$
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
75	$\frac{5\pi}{12}$	$\frac{1+\sqrt{3}}{2\sqrt{2}}$	$\frac{-1+\sqrt{3}}{2\sqrt{2}}$	$2+\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	NaN



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Arithmetic in Polar Form

- $z_1 = r_1 e^{j\theta_1} = r_1 \angle \theta_1$
- $z_2 = r_2 e^{j\theta_2} = r_2 \angle \theta_2$

- Addition
  - Convert to rectangular form to add/subtract and convert back to polar form

- Multiplication
  - Multiply magnitudes and add phase (exponent terms)
- $z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2}$
  $= r_1 r_2 e^{j(\theta_1 + \theta_2)}$
  $= r_1 r_2 \angle(\theta_1 + \theta_2)$

- Division
  - Divide magnitude and subtract phase
- $\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
  $= \frac{r_1}{r_2} \angle(\theta_1 - \theta_2)$

Euler’s Formula

- Relationship between a complex exponential and a sinusoid

- $e^{j\theta} = \cos\theta + j\sin\theta$

- $\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$
- $\sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$

Phasors

- A representation of sinusoidal signals as vectors in the complex plane
  - Simplifies sinusoidal steady-state analysis

- Given
  - $v_1(t) = V_1 \cos(\omega t + \theta_1)$
- The phasor representation is
  - $\boldsymbol{V}_1 = V_1 \angle \theta_1$
- For consistency, use only cosine for the phasor
  - $v_2(t) = V_2 \sin(\omega t + \theta_2) = V_2 \cos(\omega t + \theta_2 - 90^{\circ}) = \boldsymbol{V}_2 = V_2 \angle (\theta_2 - 90^{\circ})$

Adding Sinusoids with Phasors

1. Get phasor representation of sinusoids
2. Convert phasors to rectangular form and add
3. Simplify result and convert into phasor form
4. Convert phasor into sinusoid
    - Remember that ω should be the same for each sinusoid and the result will have the same frequency

Example 5.3

- $v_1(t) = 20\cos(\omega t - 45^\circ)$
  - $\boldsymbol{V}_1 = 20\angle(-45^\circ)$
- $v_2(t) = 10\sin(\omega t + 60^\circ)$
  - $\boldsymbol{V}_2 = 10\angle(60^\circ - 90^\circ) = 10\angle(-30^\circ)$
- Calculate
  - $\boldsymbol{V}_s = \boldsymbol{V}_1 + \boldsymbol{V}_2$

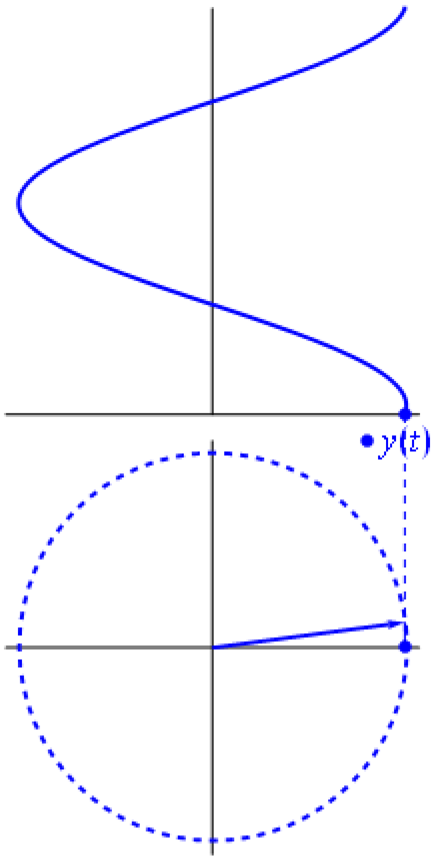
$$
\begin{aligned}
\mathbf{V}_s &= \mathbf{V}_1 + \mathbf{V}_2 \\
&= 20\underline{/-45^\circ} + 10\underline{/-30^\circ} \\
&= 20\cos(-45^\circ) + j20\sin(-45^\circ) + 10\cos(-30^\circ) + j10\sin(-30^\circ) \\
&= \frac{20}{\sqrt{2}} - j\frac{20}{\sqrt{2}} + 10\frac{sqrt3}{2} - j\frac{10}{2} \\
&= 14.14 - j14.14 + 8.66 - j5 \\
&= 22.8 - j19.14 \\
&= \sqrt{22.8^2 + 19.14^2}\underline{/\arctan\left(\tfrac{-19.14}{22.8}\right)} \\
&= 29.77\underline{/-40.01^\circ}
\end{aligned}
$$

$$v_s(t) = 29.77\cos(\omega t - 40.01^\circ)$$

Phasor as a Rotating Vector

- $v(t) = V_m \cos(\omega t + \theta) = Re\{V_m e^{j(\omega t+\theta)}\}$

- $V_m e^{j(\omega t+\theta)}$ is a complex vector that rotates counter clockwise at ω rad/s
- $v(t)$ is the real part of the vector
  - The projection onto the real axis of the rotating complex vector



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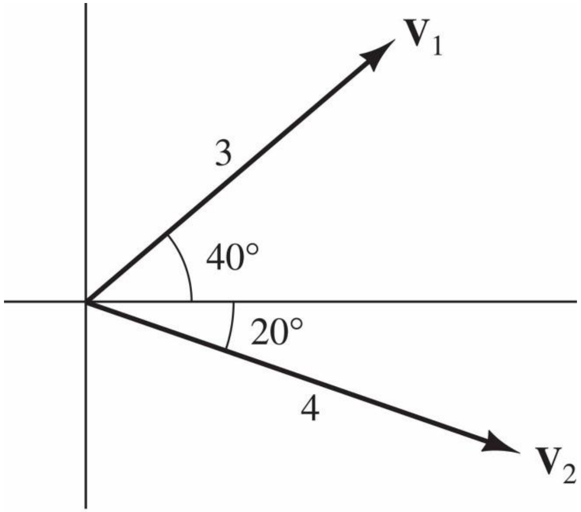
Phase Relationships

- Given
  - $v_1(t) = 3\cos(\omega t + 40^{\circ})$
  - $v_2(t) = 4\cos(\omega t - 20^{\circ})$
- In phasor notation
  - $\boldsymbol{V}_1 = 3\angle(40^{\circ})$
  - $\boldsymbol{V}_2 = 4\angle(-20^{\circ})$
- Since phasors rotate counter clockwise
  - $\boldsymbol{V}_1$ leads $\boldsymbol{V}_2$ by 60°
  - $\boldsymbol{V}_2$ lags $\boldsymbol{V}_1$ by 60°
- Phasor diagram



Phase Relationships from Plots

- Given plots of a pair of periodic waveforms

1. Find the shortest time interval t_p between positive peaks in a pair of waveforms
2. The angle between the peaks is the phase difference
   - $\theta = \frac{t_p}{T} \cdot 360^{\circ}$

Complex Impedance

- Previously we saw that resistance was a measure of the opposition to current flow
  - Larger resistance → less current allowed to flow
- Impedance is the extension of resistance to AC circuits
  - Inductors oppose a change in current
  - Capacitors oppose a change in voltage
- Capacitors and inductors have imaginary impedance → called reactance

Resistance

- From Ohm's Law
  - $v(t) = Ri(t)$
- Extend to an impedance form for AC signals
  - $\boldsymbol{V} = Z\boldsymbol{I}$
- Converting to phasor notation for resistance
  - $\boldsymbol{V}_R = R\boldsymbol{I}_R$
  - $R = \frac{\boldsymbol{V}_R}{\boldsymbol{I}_R}$
- Comparison with the impedance form results in
  - $Z_R = R$
  - Since R is real, the impedance for a resistor is purely real

Inductance

- I/V relationship
  - $v_L(t) = L\frac{di_L(t)}{dt}$
- Assume current
  - $i_L(t) = I_m \sin(\omega t + \theta)$
  - $\boldsymbol{I}_L = I_m \angle\left(\theta - \frac{\pi}{2}\right)$
- Using the I/V relationship
  - $v_L(t) = L\omega I_m \cos(\omega t + \theta)$
  - $\boldsymbol{V}_L = \omega L I_m \angle(\theta)$
  - Notice current lags voltage by 90°
- Using the generalized Ohm's Law
  - $Z_L = \frac{\boldsymbol{V}_L}{\boldsymbol{I}_L} = \frac{\omega L I_m \angle(\theta)}{I_m \angle\left(\theta - \frac{\pi}{2}\right)} = \omega L \angle\left(\frac{\pi}{2}\right) = j\omega L$
  - Notice the impedance is completely imaginary

Capacitance

- I/V relationship
  - $i_C(t) = C\frac{dv_C(t)}{dt}$
- Assume voltage
  - $v_c(t) = V_m \sin(\omega t + \theta)$
  - $\boldsymbol{V}_C = V_m \angle\left(\theta - \frac{\pi}{2}\right)$
- Using the I/V relationship
  - $i_C(t) = C\omega V_m \cos(\omega t + \theta)$
  - $\boldsymbol{I}_C = \omega C V_m \angle(\theta)$
  - Notice current leads voltage by 90°
- Using the generalized Ohm's Law
  - $Z_C = \frac{\boldsymbol{V}_C}{\boldsymbol{I}_C} = \frac{V_m \angle\left(\theta - \frac{\pi}{2}\right)}{\omega C V_m \angle(\theta)} = \frac{1}{\omega C}\angle\left(-\frac{\pi}{2}\right) = \frac{1}{j\omega C}$
  - Notice the impedance is completely imaginary

Circuit Analysis with Impedance

- KVL and KCL remain the same
  - Use phasor notation to setup equations

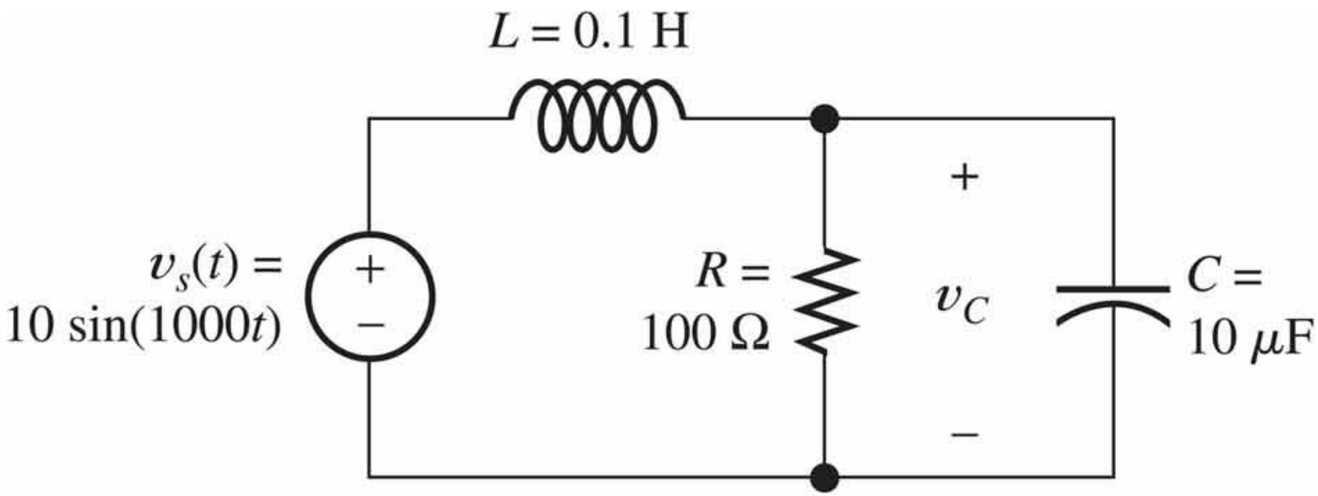
- E.g.
- $v_1(t) + v_2(t) - v_3(t) = 0$
- $\boldsymbol{V}_1 + \boldsymbol{V}_2 - \boldsymbol{V}_3 = 0$

Steps for Sinusoidal Steady-State Analysis

1. Replace time descriptions of voltage and current sources with corresponding phasors. (All sources must have the same frequency)
2. Replace inductances by their complex impedances $Z_L = \omega L \angle \left(\frac{\pi}{2}\right) = j\omega L$. Replace capacitances by their complex impedances $Z_C = \frac{1}{\omega C} \angle \left(-\frac{\pi}{2}\right) = \frac{1}{j\omega C}$. Resistances have impedances equal to their resistances.
3. Analyze the circuit by using any of the techniques studied in Chapter 2, and perform the calculations with complex arithmetic

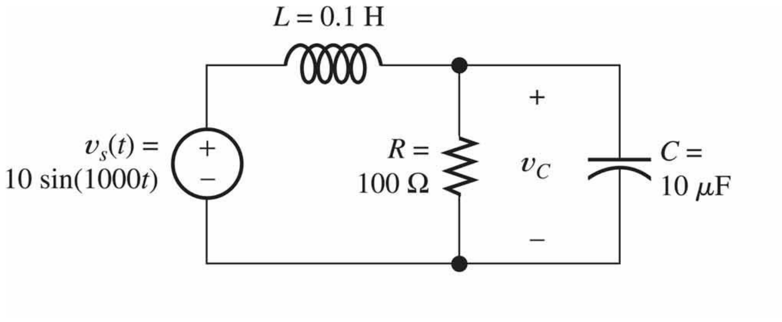
Example 5.5

- Find voltage $v_c(t)$ in steady-state
- Find the phasor current through each element
- Construct the phasor diagram showing currents and v_s



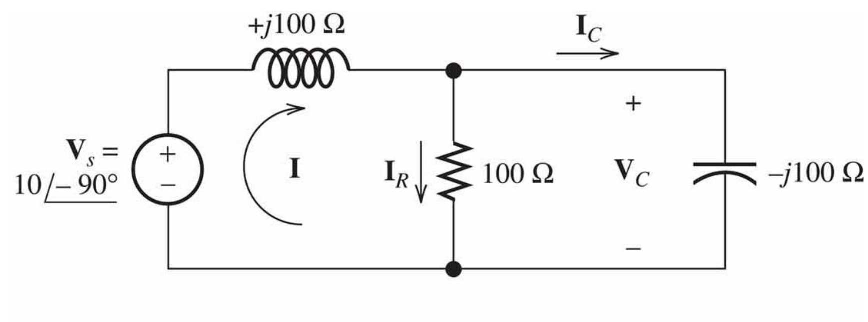
Convert Sources and Impedances

- Voltage source
  - $\boldsymbol{V}_s = 10\angle(-90^{\circ})$
  - $\omega = 1000$

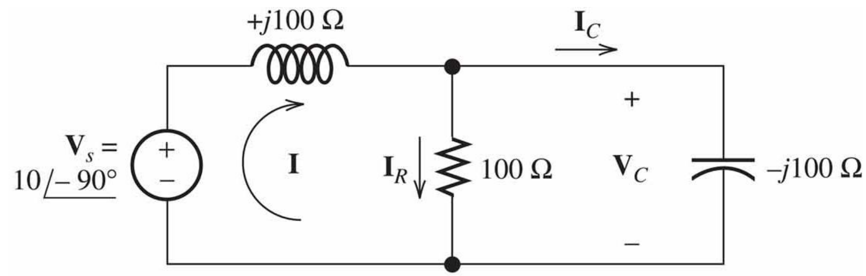


- Inductance
  - $Z_L = j\omega L = \text{j}(1000)(0.1) = \text{j}100\ \Omega$

- Capacitance
  - $Z_C = \frac{1}{j\omega C} = \frac{1}{j1000(10\mu)} = \frac{10^6}{j10^4} = \frac{100}{j}\ \Omega$



Find Voltage $v_c(t)$



- Find voltage by voltage divider
- $V_C = V_s\left(\frac{Z_{eq}}{Z_{eq}+Z_L}\right)$

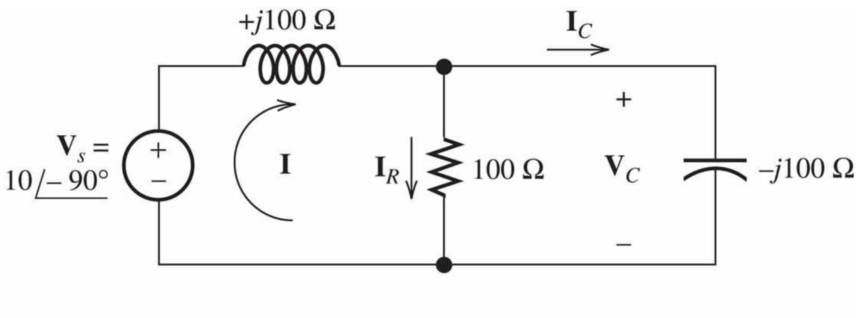
$$\mathbf{V}_C = \mathbf{V}_s\left(\frac{Z_{eq}}{Z_{eq}+Z_L}\right)$$
$$= 10\angle -90^\circ \left(\frac{\frac{100}{\sqrt{2}}\angle -45^\circ}{50 - j50 + j100}\right)$$
$$= 10\angle -90^\circ \left(\frac{\frac{100}{\sqrt{2}}\angle -45^\circ}{50 + j50}\right)$$
$$= 10\angle -90^\circ \left(\frac{\frac{100}{\sqrt{2}}\angle -45^\circ}{\frac{100}{\sqrt{2}}\angle -45^\circ}\right)$$
$$= 10\angle -180^\circ$$

- Equivalent impedance
- $Z_{eq} = Z_R || Z_C$

$$Z_{eq} = \frac{Z_C Z_R}{Z_C + Z_R} = \frac{100\angle -\pi/2 (100\angle 0)}{-j100 + 100}$$
$$= \frac{100^2\angle 90^\circ}{100\sqrt{2}\angle 45^\circ} = \frac{100}{\sqrt{2}}\angle -45^\circ$$
$$= 70.71\angle -45^\circ$$
$$= 50 - j50$$

$$v_c(t) = 10\cos(\omega t - 180^\circ) = 10\cos(1000t - \pi) = -10\cos(1000t)$$

Phasor Diagram



- Source current

$$\mathbf{I} = \frac{\mathbf{V}_s}{Z_{eq} + Z_L} = \frac{10\underline{/-90^\circ}}{100/\sqrt{2}\underline{/45^\circ}} = \frac{\sqrt{2}}{10}\underline{/-135^\circ}$$

- Capacitor current

$$\mathbf{I}_C = \frac{\mathbf{V}_C}{Z_C} = \frac{10\underline{/-180^\circ}}{100\underline{/-90^\circ}} = \frac{1}{10}\underline{/-90^\circ}$$

- Resistor current

$$\mathbf{I}_R = \frac{\mathbf{V}_C}{Z_R} = \frac{10\underline{/-180^\circ}}{100\underline{/0^\circ}} = \frac{1}{10}\underline{/-180^\circ}$$

- Phasor diagram