Outline

- Review
  - 1\textsuperscript{st}-Order Transients
  - General Sources
  - 2\textsuperscript{nd}-Order Circuits
General 1\textsuperscript{st}-Order Solution

- Both the current and voltage in an 1\textsuperscript{st}-order circuit has an exponential form
  - RC and LR circuits

- The general solution for current/voltage is:
  \[ x(t) = x_f + \left[ x(t_0^+) - x_f \right] e^{-(t-t_0)/\tau} \]
  - \( x \) – represents current or voltage
  - \( t_0 \) – represents time when source switches
  - \( x_f \) - final (asymptotic) value of current/voltage
  - \( \tau \) – time constant (\( RC \) or \( \frac{L}{R} \))

- Find values and plug into general solution
RC/RL Circuits with General Sources

• Previously,
  \[ RC \frac{dv_c(t)}{dt} + v_c(t) = V_s \]

• What if \( V_s \) is not constant
  \[ RC \frac{dv_c(t)}{dt} + v_c(t) = v_s(t) \]
  Now have a general source that is a function of time

• The solution is a differential equation of the form
  \[ \tau \frac{dx(t)}{dt} + x(t) = f(t) \]
  Where \( f(t) \) is known as the forcing function (the circuit source)
General Differential Equations

- General differential equation
  \[ \tau \frac{dx(t)}{dt} + x(t) = f(t) \]
- The full solution to the diff equation is composed of two terms
  \[ x(t) = x_p(t) + x_h(t) \]
- \( x_p(t) \) is the particular solution
  - The response to the particular forcing function
- \( x_h(t) \) is the homogeneous solution
  - Another solution that is consistent with the differential equation for \( f(t) = 0 \)
  - The response to any initial conditions of the circuit
Particular Solution

- \( \tau \frac{dx_p(t)}{dt} + x_p(t) = f(t) \)
- The solution \( x_p(t) \) is called the forced response because it is the response of the circuit to a particular forcing input \( f(t) \)
- The solution \( x_p(t) \) will be of the same functional form as the forcing function
  - E.g.
    - \( f(t) = e^{st} \rightarrow x_p(t) = Ae^{st} \)
    - \( f(t) = \cos(\omega t) \rightarrow x_p(t) = A\cos(\omega t) + B\sin(\omega t) \)
Homogeneous Solution

\[ \tau \frac{dx_h(t)}{dt} + x_h(t) = 0 \]

- \( x_h(t) \) is the solution to the differential equation when there is no forcing function
- Does not depend on the sources
- Dependent on initial conditions (capacitor voltage, current through inductor)

- \( x_h(t) \) is also known as the natural response

- Solution is of the form
  - \( x_h(t) = Ke^{-t/\tau} \)
General Differential Solution

• Notice the final solution is the sum of the particular and homogeneous solutions
  \[ x(t) = x_p(t) + x_h(t) \]
• It has an exponential term due to \( x_h(t) \) and a term \( x_p(t) \) that matches the input source
Second-Order Circuits

• RLC circuits contain two energy storage elements
  ▫ This results in a differential equation of second order (has a second derivative term)

• This is like a mass spring system from physics
RLC Series Circuit

- **KVL around loop**
  - $v_s(t) - L \frac{di(t)}{dt} - i(t)R - v_c(t) = 0$
- **Solve for $v_c(t)$**
  - $v_c(t) = v_s(t) - L \frac{di(t)}{dt} - i(t)R$
- **Take derivative**
  - $\frac{dv_c(t)}{dt} = \frac{dv_s(t)}{dt} - L \frac{d^2i(t)}{dt^2} - R \frac{di(t)}{dt}$

- **Solve for current through capacitor**
  - $i(t) = C \frac{dv_c(t)}{dt}$
  - $i(t) = C \left[ \frac{dv_s(t)}{dt} - L \frac{d^2i(t)}{dt^2} - R \frac{di(t)}{dt} \right]$
  - $\frac{d^2i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{1}{L} \frac{dv_s(t)}{dt}$

  - **The general 2nd-order constant coefficient equation**

$$\frac{di^2(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)$$

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad f(t) = \frac{1}{L} \frac{dv_s(t)}{dt}$$
Circuits with Parallel L and C

- Given a circuit with a parallel capacitor and inductor
  - Use Norton equivalent to make a parallel circuit equivalent
Circuits with Parallel L and C

- Find current in inductor $i_L(t)$
- KCL @ top node
- $i_n(t) = \frac{v(t)}{R} + i_L(t) + C \frac{dv(t)}{dt}$

- $i_n(t) = \frac{L}{R} \frac{di(t)}{dt} + i(t) + CL \frac{d^2i(t)}{dt^2}$
- $\frac{d^2i(t)}{dt^2} + \frac{1}{RC} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{1}{CL} i_n(t)$

- $\frac{di^2(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)$

- $f(t) = \frac{1}{LC} i_n(t)$
- $\alpha = \frac{1}{2RC}$
- $\omega_0 = \sqrt{\frac{1}{LC}}$
Circuits with Parallel L and C

- Find parallel voltage \( v(t) \)
- KCL @ top node
  \[
i_n(t) = \frac{v(t)}{R} + i_L(t) + C \frac{dv(t)}{dt}
\]
- \( i_n(t) = \frac{v(t)}{R} + \frac{1}{L} \int_{t_0}^{t} v(t) \, dt + i(t_0) + C \frac{dv(t)}{dt} \)
- Take derivative of both sides with respect to time
  \[
  \frac{di_n(t)}{dt} = \frac{1}{R} \frac{dv(t)}{dt} + \frac{1}{L} v(t) + C \frac{d^2v(t)}{dt^2}
  \]
  \[
  \frac{d^2v(t)}{dt^2} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = \frac{1}{C} \frac{di_n(t)}{dt}
  \]
- \( f(t) = \frac{1}{C} \frac{di_n(t)}{dt} \)
- \( \alpha = \frac{1}{2RC} \)
- \( \omega_0 = \sqrt{\frac{1}{LC}} \)
- \( v(t) = L \frac{di(t)}{dt} \)
- \( i(t) = \frac{1}{L} \int_{t_0}^{t} v(t) \, dt + i(t_0) \)
  \[
  \frac{di^2(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)
  \]