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ECG782: Multidimensional Digital Signal Processing

Spring 2014
TTh 14:30-15:45 CBC C313

Lecture 05
Image Processing Basics
13/02/04

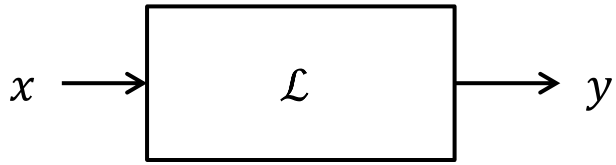
http://www.ee.unlv.edu/~b1morris/ecg782/

Outline

- Linear Systems
- 1D Fourier Review
- 2D Fourier Transform
- Discrete Cosine Transform
- Wavelet Transform

Linearity

- Image processing as a system



- Linearity defined on vector (linear) spaces
  - Superposition principle:
    - Additivity
    - Homogeneity
  - $\mathcal{L}(ax_1 + bx_2) = a\mathcal{L}(x_1) + b\mathcal{L}(x_2)$

- Very important property for linear image processing
  - Can make use of 1D signal processing ideas

Dirac Delta Distribution

- Dirac delta - $\delta(x, y)$
  - Impulse - $\delta(x, y) = 0 \; \forall (x, y) \neq 0$
  - $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) dxdy = 1$
- Sifting property
  - Use of $\delta(x, y)$ to obtain value of function
  - $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\delta(x - \lambda, y - \mu) dxdy = f(\lambda, \mu)$
- Delta sampling
  - $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a, b)\delta(a - x, b - y) dadb = f(x, y)$

Convolution

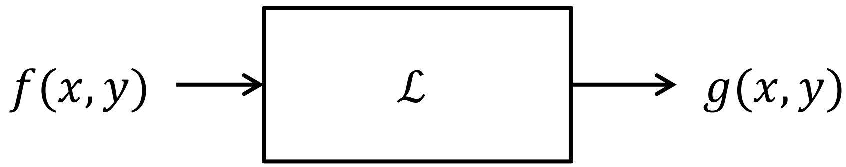
- In image processing, this is an "overlap" operation
- 1D convolution
  - $(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)f(t-\tau)d\tau$
  - Limits restricted to finite support (similar to image range)
- Properties
  - $f * h = h * f$
  - $f * (g * h) = (f * g) * h$
  - $f * (g + h) = f * g + f * h$
  - $a(f * g) = (af) * g = f * (ag)$
  - $\frac{d}{dx}(f * h) = \frac{df}{dx} * h = f * \frac{dh}{dx}$

- 2D convolution
- $(f * h)(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a, b)h(x-a, y-b)dadb$
- $= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(a, b)f(x-a, y-b)dadb$
- $= (h * f)(x, y)$

- Discrete 2D convolution
  - Linear preprocessing step
  - Output pixel is a linear combination of neighborhood pixels
- $g(i, j) = \sum_{(m,n)\in\mathcal{O}} h(i-m, j-n)f(m, n)$
  - $\mathcal{O}$ – neighborhood (typically rectangular with odd number or rows and columns
  - h - kernel or convolution mask

Images as Linear Systems

- Image viewed as superposition of deltas



- $g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a,b)\mathcal{L}\{\delta(a-x,b-y)\}dadb$
- $g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a,b)h(a-x,b-y)dadb$
- $g(x,y) = (f * h)(x,y)$
- Fourier transform relationship
- $G(u,v) = F(u,v)H(u,v)$
  - Useful for frequency domain smoothing and sharpening

Intro Linear Integral Transforms

- Represent signals (images) in a more suitable domain
  - Information is better “visible”
  - Solution to related (dual) problem is easier
- Most often interested in the “frequency domain”
  - What a one-to-one mapping between spatial (image coordinates) and frequency
    - Inverse transform must exist
  - Popular transforms include:
    - Fourier, Cosine, Wavelet
- Most often used for filtering
  - Image-to-image mapping

1D Fourier Transform

- $F\{f(t)\} = F(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}dt$
- Inverse transform
  - $F^{-1}\{F(\xi)\} = f(t) = \int_{-\infty}^{\infty} F(\xi)e^{2\pi i\xi t}d\xi$
- This always exists for digital signals (of finite length, e.g. images)
- Notice the FT is a linear combination of complex exponentials
  - Linear combination of sines and cosines
  - $e^{j\theta} = \cos\theta + j\sin\theta$
  - $F(\xi)$ – indicates the contribution of sinusoid with frequency ξ

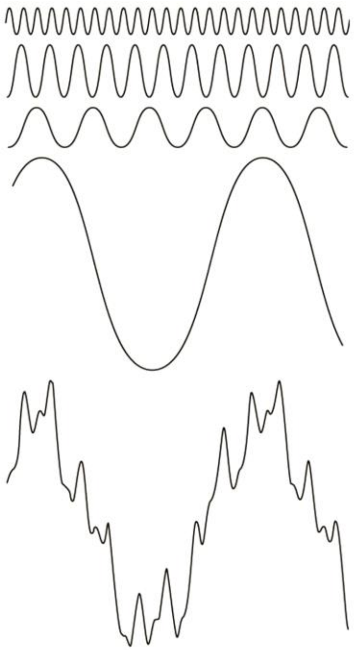


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

1D Fourier Transform Representation

- In general, FT is complex valued
- Can express in polar form
  - $F(\xi) = |F(\xi)|e^{-i\phi(\xi)}$
- Magnitude spectrum
  - $|F(\xi)| = [Re^2(F) + Im^2(F)]^{1/2}$
- Phase spectrum (angle)
  - $\phi(\xi) = \tan^{-1}\left[\frac{Im(F)}{Re(F)}\right]$
- Power spectrum
  - $P(\xi) = |F(\xi)|^2 = Re^2(F) + Im^2(F)$

Fourier Transform Properties

- DC value (offset)
  - $F(0) = \int_{-\infty}^{\infty} f(t)dt$
  - Area under $f(t)$
  - Average value
- FT offset
  - $f(0) = \int_{-\infty}^{\infty} F(\xi)d\xi$
- Parseval's Theorem
  - $\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\xi)|^2 d\xi$
  - Energy in time domain is equal to energy in frequency domain
- Uncertainty principle
  - Wide in time → narrow in frequency
  - Narrow in time → wide in frequency

Property	$f(t)$	$F(\xi)$
Linearity	$af_1(t) + bf_2(t)$	$a\,F_1(\xi) + b\,F_2(\xi)$
Duality	$F(t)$	$f(-\xi)$
Convolution	$(f * g)(t)$	$F(\xi)\,G(\xi)$
Product	$f(t)\,g(t)$	$(F * G)(\xi)$
Time shift	$f(t - t_0)$	$e^{-2\pi i \xi t_0}\,F(\xi)$
Frequency shift	$e^{2\pi i \xi_0 t} f(t)$	$F(\xi - \xi_0)$
Differentiation	$\frac{\mathrm{d}f(t)}{\mathrm{d}t}$	$2\pi i \xi F(\xi)$
Multiplication by t	$t\,f(t)$	$\frac{i}{2\pi}\frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi}$
Time scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F(\xi/a)$

Table 3.3: Properties of the Fourier transform. © *Cengage Learning 2015.*

Short Time Fourier Transform

- Non-stationary signal processing technique
  - Signal distribution changes in time
- Divide signal into smaller pieces and do computations in windows
  - Typically, smooth windows are selected to reduce border effects

- This gives a sense of "global" properties but also a time when they happen (which window)
  - Global – mean, variance, frequency content
  - Timing – which window

Discrete Fourier Transform

- Assume discrete signal obtained by sampling
  - $f(n),\ n = 0 \ldots N-1$
- DFT
  - $F(k) = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\exp(-2\pi i\frac{nk}{N})$
  - $f(n) = \sum_{k=0}^{N-1} F(k)\exp(2\pi i\frac{nk}{N})$
- Since DFT is discrete, it is periodic
  - k – represents a discrete frequency
- Fast Fourier transform (FFT)
  - Fast implementation of DFT ($O(n\log n)$)
  - Basic DFT is $O(n^2)$
  - Makes frequency domain processing possible

2D Fourier Transform

- Generalization of 1D FT
- $F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-2\pi i(xu+yv)}\, dxdy$
- $f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v)e^{2\pi i(xu+yv)}\, dudv$
- For images (u, v) are called spatial frequencies
  - FT indicates how to combine 2D spatial sinusoids

- Properties:
- Linearity
  - $F\{af_1(x, y) + bf_2(x, y)\} = aF_1(u, v) + bF_2(u, v)$
- Time (spatial) shift
  - $F\{f(x - a, y - b)\} = F(u, v)e^{-2\pi i(au+bv)}$
- Frequency shift
  - $F\left\{f(x, y)e^{2\pi i(u_0 x+v_0 y)}\right\} = F(u - u_0, v - v_0)$
- Real $f(x, y)$
  - $F(-u, -v) = F^*(u, v)$
  - Only need first quadrant for images $(u \geq 0, v \geq 0)$
- Convolution duality
  - $F\{(f * h)(x, y)\} = F(u, v)H(u, v)$
  - $F\{f(x, y)h(x, y)\} = (F * H)(u, v)$

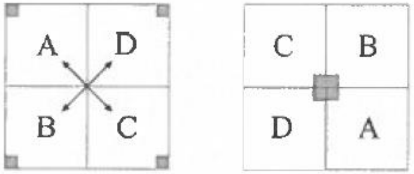
Discrete 2D Fourier Transform

- $F(u,v) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \exp\left[-2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)\right]$
  - $u = 0,1, \dots M-1, \quad v = 0,1, \dots, N-1$
- $f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp\left[2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)\right]$
  - $m = 0,1, \dots M-1, \quad n = 0,1, \dots, N-1$
- Efficient implementation with 1D FFT
  - Compute FFT of each row m
  - Compute FFT of each column n (of FFT coefficients)
- Notice this is a period function
  - Periodic in two directions
    - v direction: period N, $\Delta u = 1/M\Delta x$
    - u direction: period M, $\Delta v = 1/N\Delta y$

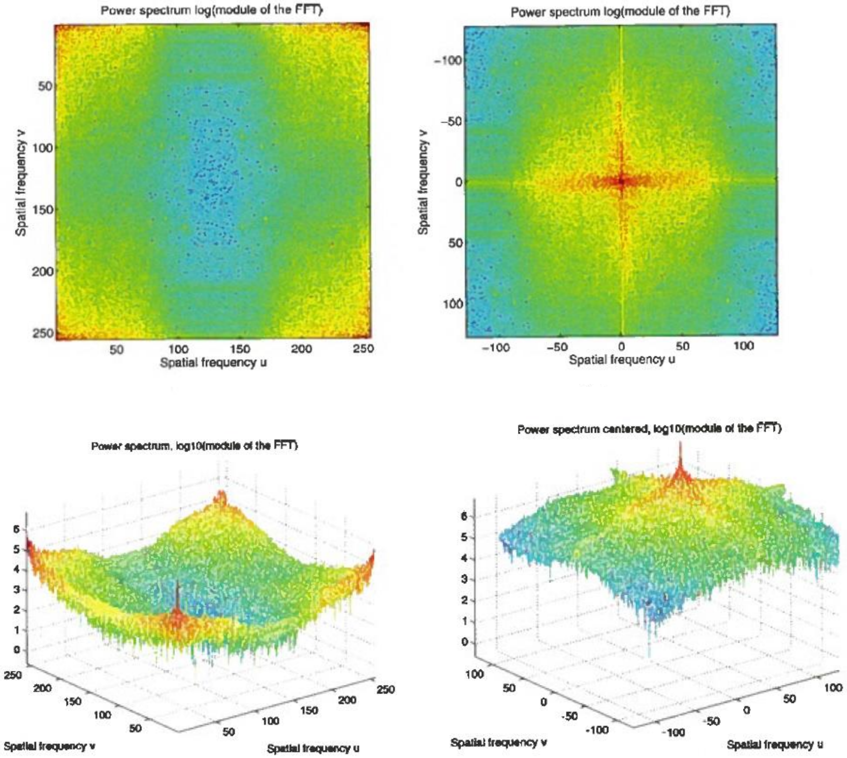
2D Fourier Transform Example

- Input image
  - Assumed periodicity for harmonic frequencies (discrete)

- Remember that origin is typically in the top right
  - Low frequency components are in the corners of FT image

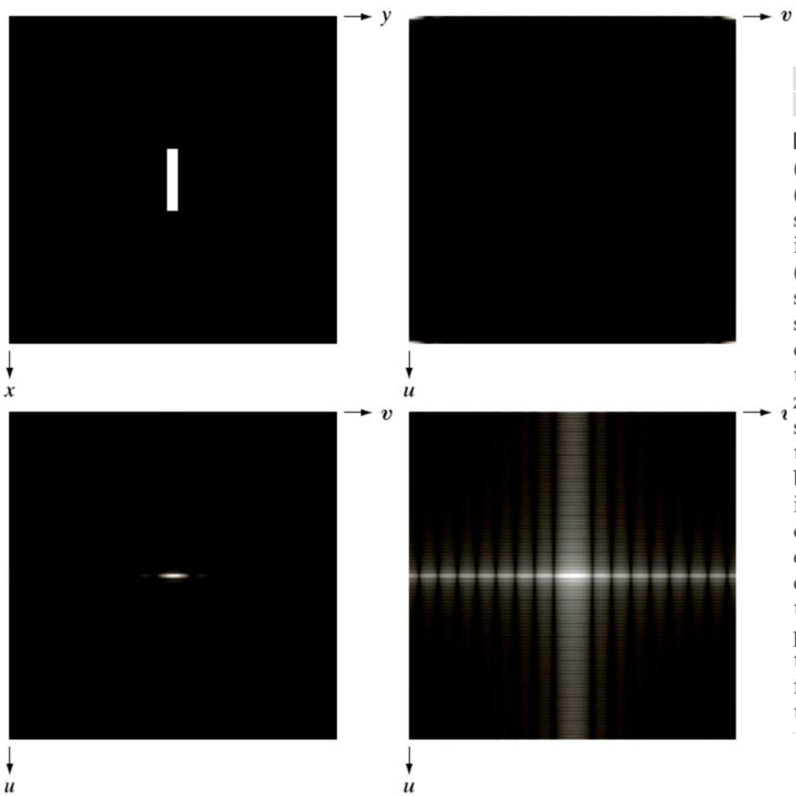


(a) (b)



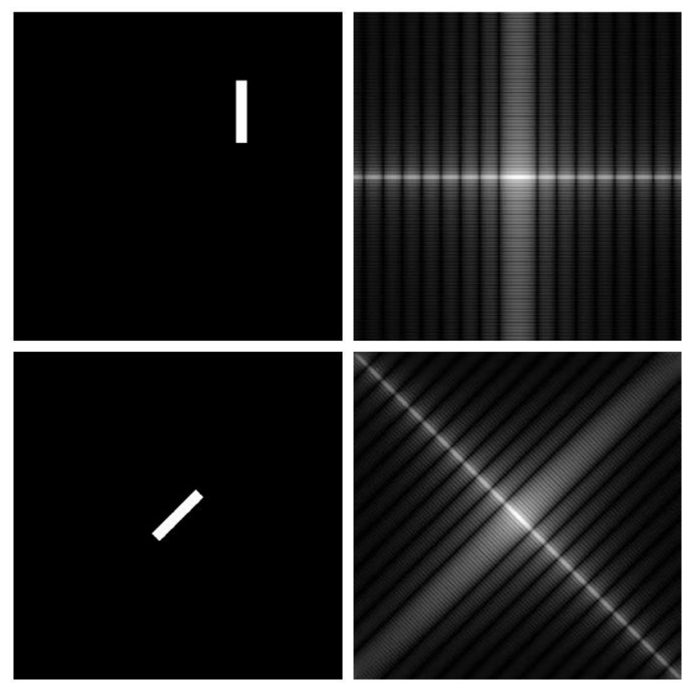
(a) (b)

More Fourier Transform Examples



a b
c d

FIGURE 4.24
(a) Image.
(b) Spectrum showing bright spots in the four corners.
(c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

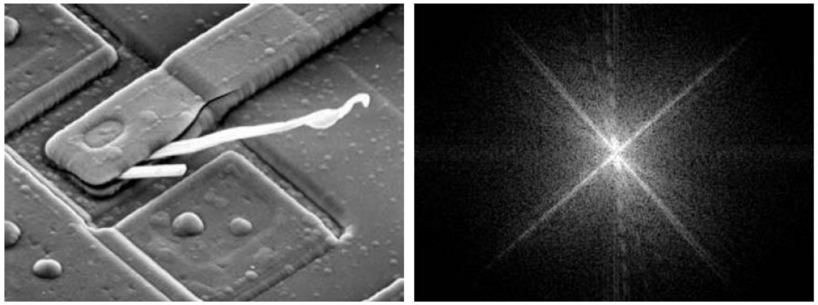


a b
c d

FIGURE 4.25
(a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum.
(c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

More Fourier Transform Examples

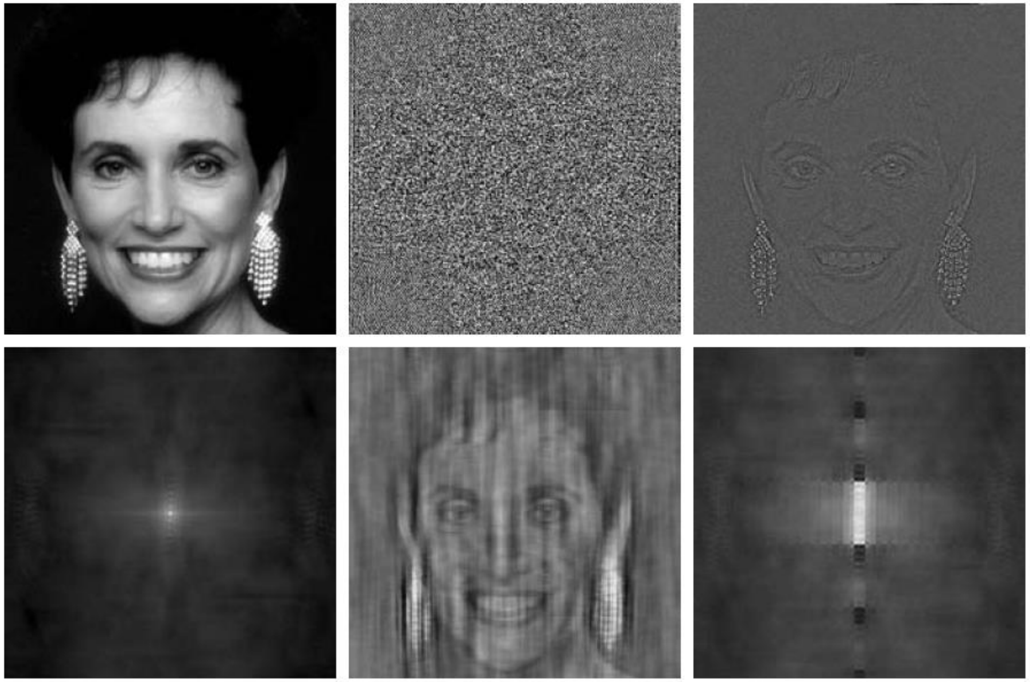
- Importance of magnitude and phase



a b

FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

- Higher values for edges and changing textures
  - Notice the 45 degree line



a b c
d e f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

Sampling

- Sample the continuous image function
- Sampling function
  - $s(x,y) = \sum_{j=1}^{M} \sum_{k=1}^{N} \delta(x - j\Delta x, y - k\Delta y)$
    - $\Delta x, \Delta y$ – sampling intervals
- Sampling signal
- $f_s(x,y) = f(x,y)s(x,y)$
  - $= f(x,y) \sum_{j=1}^{M} \sum_{k=1}^{N} \delta(x - j\Delta x, y - k\Delta y)$
- Taking FT of both sides
  - $F_s(u,v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(u - \frac{m}{\Delta x}, v - \frac{n}{\Delta y})$
  - Repeated copies of $F(u,v)$ (DTFT)

Shannon's Sampling Theorem

- Periodic copies of spectrum can result in image distortion (aliasing)
  - Occurs when copies overlap
  - Caused by undersampling
- Shannon's sampling theorem
  - $\Delta x < \frac{1}{2U}, \ \Delta y < \frac{1}{2V}$
  - U, V – max frequencies in image
  - Sampling interval should be less than half the smallest image detail

- In reality, sampling grid is used
- $f_s(x,y) = \sum_{j=1}^{M} \sum_{k=1}^{N} f(x,y) h_s(x - j\Delta x, y - k\Delta y)$
- $F_s(u,v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F\left(u - \frac{m}{\Delta x}, v - \frac{n}{\Delta y}\right) \cdot \mathrm{H_s}\left(\frac{m}{\Delta x}, \frac{n}{\Delta y}\right)$

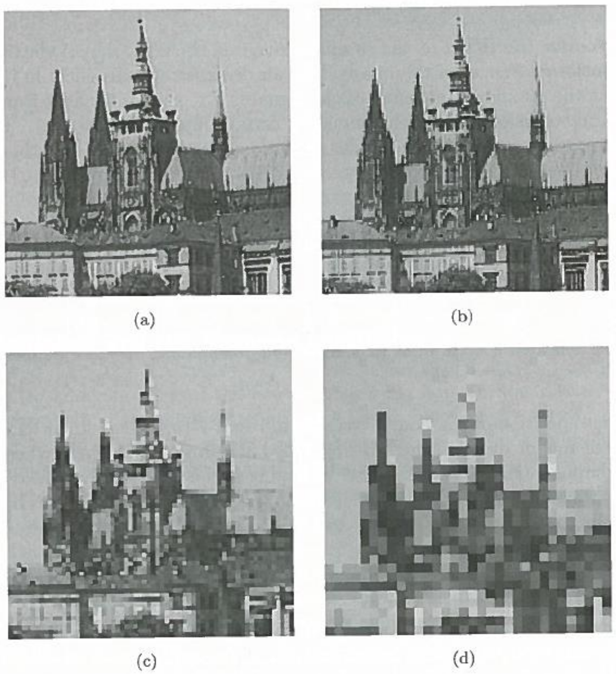


Figure 3.11: Digitizing. (a) 256 × 256. (b) 128 × 128. (c) 64 × 64. (d) 32 × 32. Images have been enlarged to the same size to illustrate the loss of detail. © *Cengage Learning 2015.*

Discrete Cosine Transform

- Similar to DFT but not complex
  - Double length DFT with even functions
- Four basic DCT types depending on type of periodic extension applied at boundaries
  - DCT-I, -II, -III, -IV
- Image processing uses DCT-II (compression, object detection/recognition)
  - Even extension at both left and right boundaries
  - Mirroring results in smooth period function which requires less coefficients for approximation

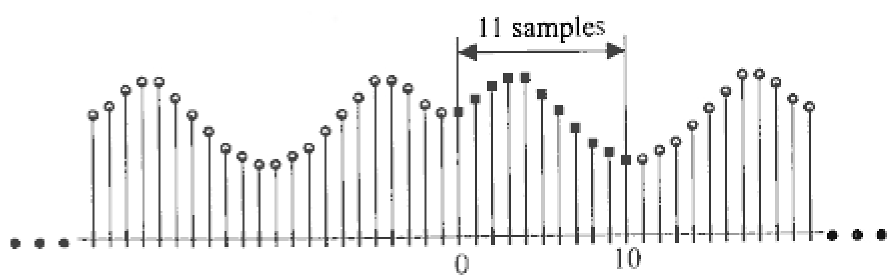


Figure 3.12: Illustration of the periodic extension used in DCT-II. The input signal of length 11 is denoted by squares. Its periodic extension is shown as circles. © *Cengage Learning 2015.*

2D DCT

- $F(u,v) = \frac{2c(u)c(v)}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m,n)\cos\left(\frac{2m+1}{2N}u\pi\right)\cos\left(\frac{2n+1}{2N}v\pi\right)$
  - $u - 0,1,\ldots N-1, v = 0,1,\ldots N-1$
  - $c(k) = \begin{cases} 1/\sqrt{2} & k = 0 \\ 1 & else \end{cases}$
- For highly correlated images, is able to compact energy into fewer coefficients
  - Useful for compression (image, video)
    - Used in JPEG, MPEG-4
- Similar to DFT
  - Can use FFT type calculations for speed
  - DC is zeroth component

2D DCT Example

- Comparison with DFT

a
b

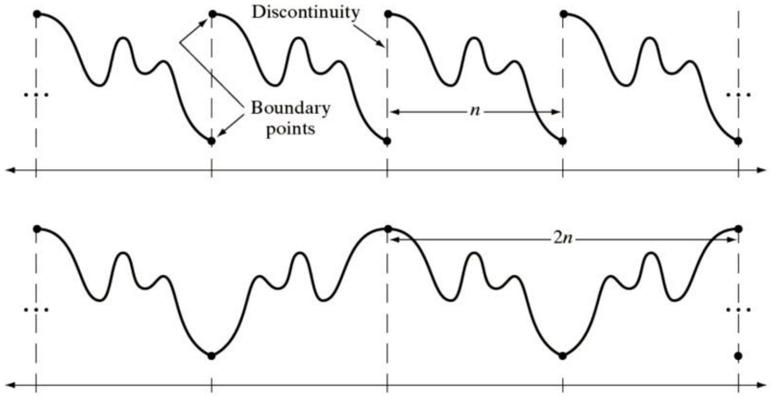


FIGURE 8.25 The periodicity implicit in the 1-D (a) DFT and (b) DCT.

- DCT basis

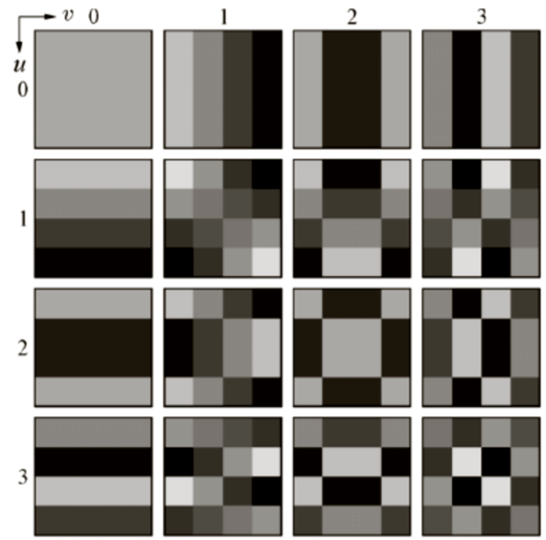
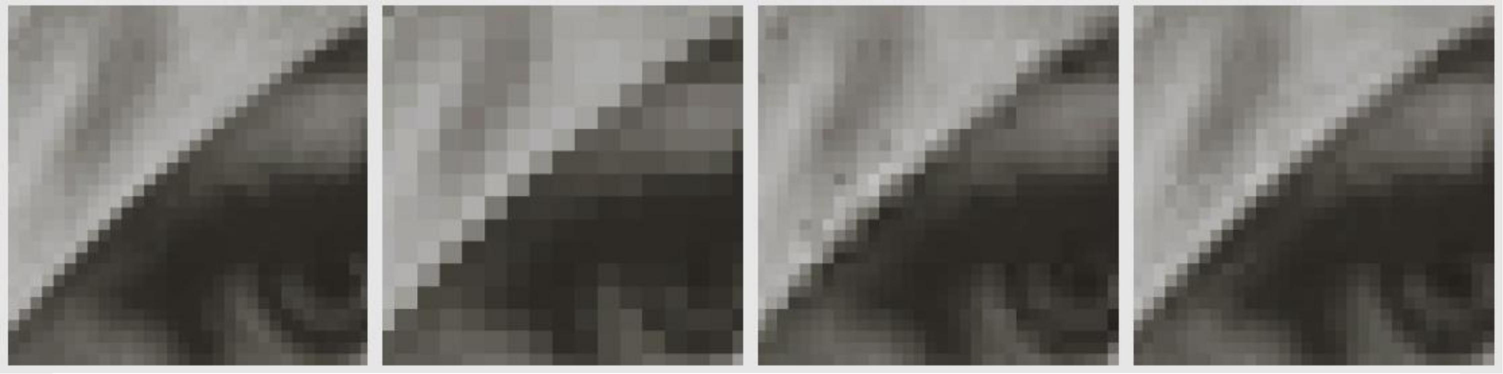


FIGURE 8.23 Discrete-cosine basis functions for $n = 4$. The origin of each block is at its top left.

- Subwindow size



a b c d

FIGURE 8.27 Approximations of Fig. 8.27(a) using 25% of the DCT coefficients and (b) 2×2 subimages, (c) 4×4 subimages, and (d) 8×8 subimages. The original image in (a) is a zoomed section of Fig. 8.9(a).

Wavelet Transform

- Decompose signals as linear combination of another set of basis functions (not sinusoid)
  - Can be more complex basis
- Mother wavelets

Figure 3.13: Examples of mother wavelets. © *Cengage Learning 2015.*

- Multiscale analysis
  - Provide localization in space
  - Search for particular "pattern" a different scales
- Wavelets are better designed for digital images
  - Less coefficients required than for sinusoidal
    - Think about how many coefficients are required for a single on pixel (delta)

1D Continuous Wavelet Transform

- $c(s,\tau) = \int_R f(t)\Psi^*_{s,\tau}(t)dt$
  - $s \in R^+ - \{0\}$ – indicates scale
  - $\tau \in R$ – indicates a time shift
- Wavelets at scale and shift generated from a "mother" wavelet
  - $\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right)$
- Wavelet functions must have two properties
  - Admissibility – must have bandpass spectrum
    - Use oscillatory functions
  - Regularity – must have smoothness and concentration in time/frequency domains
    - Fast decrease with decreasing scale

Haar Wavelet

- "Mother" function (basis)
  - $\Psi_{ji}(x) = 2^{\frac{j}{2}}\Psi(2^j x - i)$
  - $\Psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & else \end{cases}$

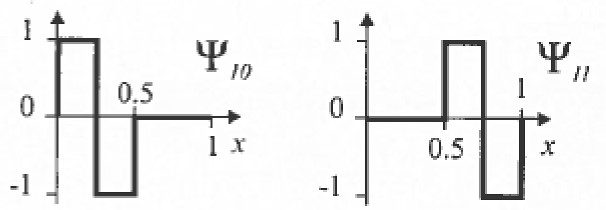


Figure 3.15: Haar wavelets Ψ_{11}, Ψ_{12}. © *Cengage Learning 2015.*

- Scaling ("Father") function (multi-resolution/scale)
  - $\Phi_{ji}(x) = 2^{\frac{j}{2}}\Phi(2^j x - i)$
  - $\Phi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & else \end{cases}$
  - Scaled and translated box functions

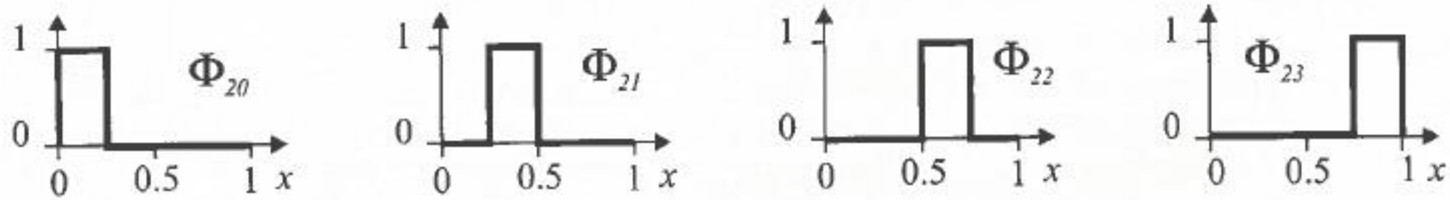


Figure 3.14: 'Box-like' scaling functions Φ. © *Cengage Learning 2015.*

Discrete Wavelet Transform

- Computationally efficient implementation
  - Herringbone algorithm exploits relationship between coefficients at various scales
- 1D case:
- At each level produce approximation coefficients and details
  - Approximation from lowpass
  - Detail from highpass
  - Use downsample to change scale
- Better approximation with more coefficients (more levels/scale)

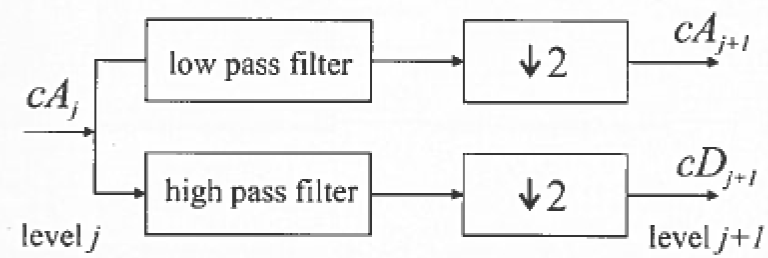


Figure 3.16: A single decomposition step of the 1D discrete wavelet transform consists of the convolution of coefficients from previous level j by a low/high pass filter and down-sampling by dyadic decimation. Approximation and detail coefficients at level $j+1$ are obtained. © *Cengage Learning 2015.*

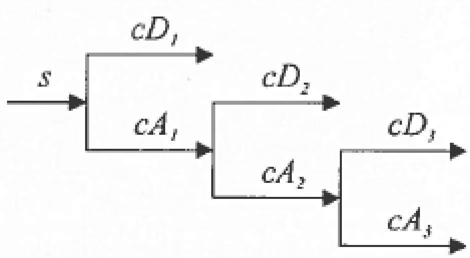


Figure 3.17: Example illustrating the structure of approximation and detail coefficients for levels up to a level $j = 3$. © *Cengage Learning 2015.*

2D Wavelet Transform

- Similar idea and extension from 1D to 2D
- 2D case
  - 4 decomposition types
    - Approximation
    - 3 detail – horizontal, vertical, and diagonal

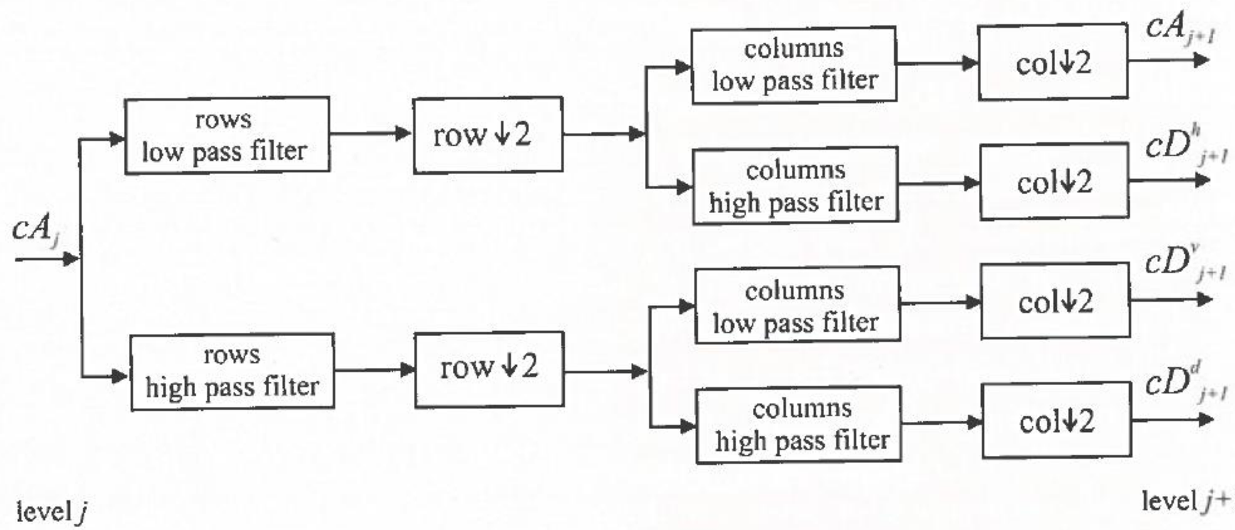


Figure 3.19: 2D discrete wavelet transform. A decomposition step. © *Cengage Learning 2015.*

2D Wavelet Transform Example

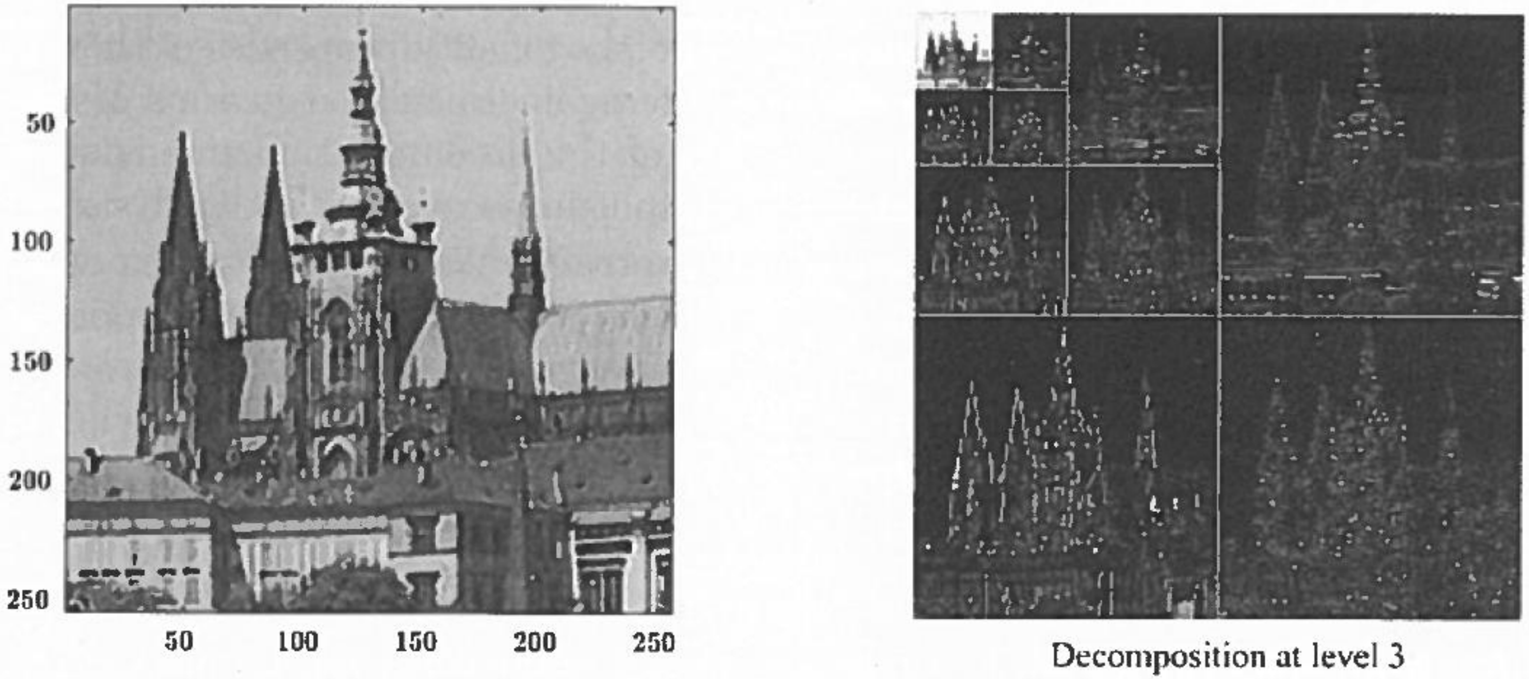


Figure 3.21: Decomposition to three levels by the 2D discrete Haar wavelet transform. Left is the original 256 × 256 gray-scale image, and right four quadrants. The undivided southwestern, southeastern and northeastern quadrants correspond to detailed coefficients of level 1 at resolution 128 × 128 in vertical, diagonal and horizontal directions, respectively. The northwestern quadrant displays the same structure for level 2 at resolution 64 × 64. The northwestern quadrant of level 2 shows the same structure at level 3 at resolution 32 × 32. The lighter intensity 32 × 32 image at top left corresponds to approximation coefficients at level 3. © *Cengage Learning 2015*.

2D Wavelet Transform Example

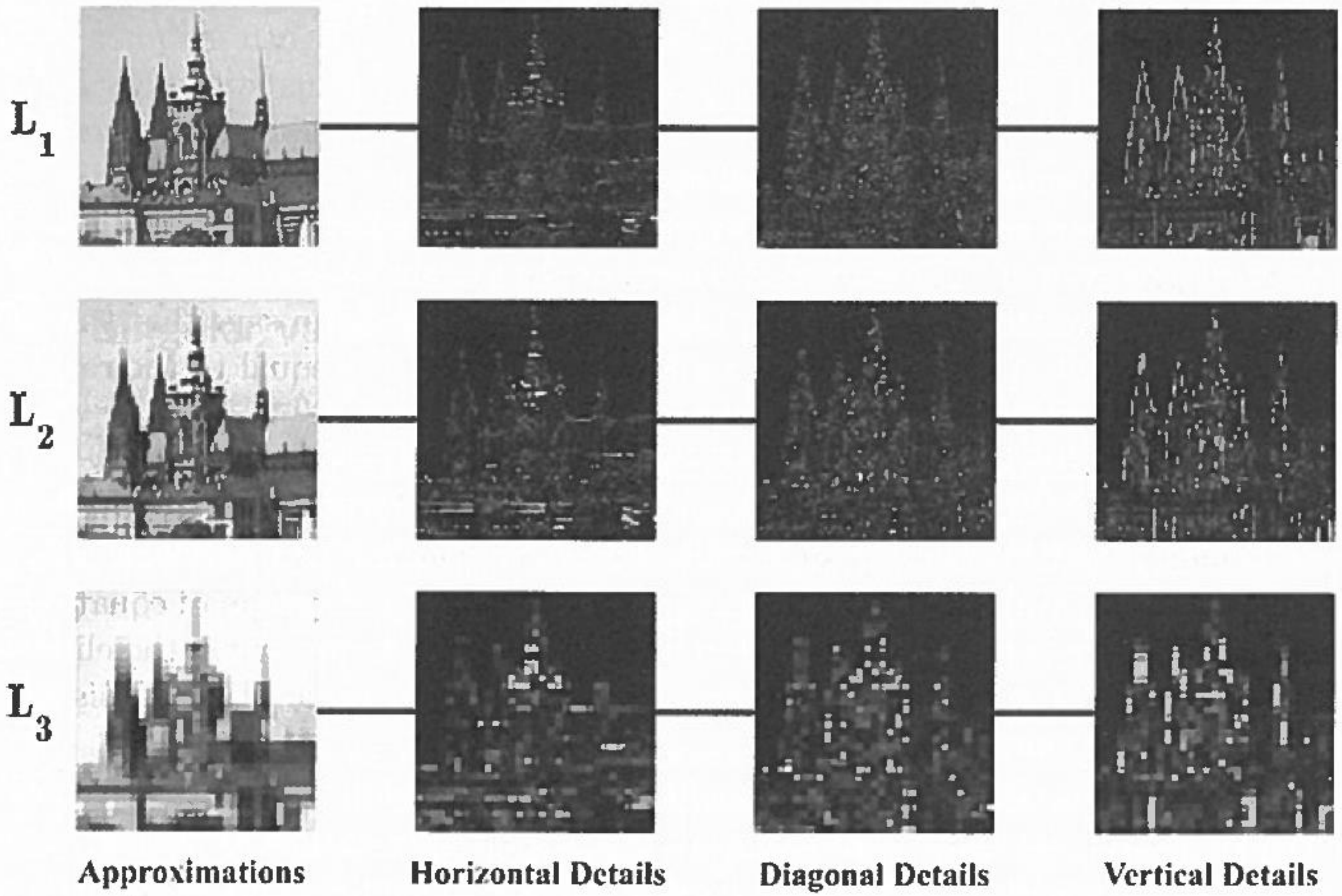
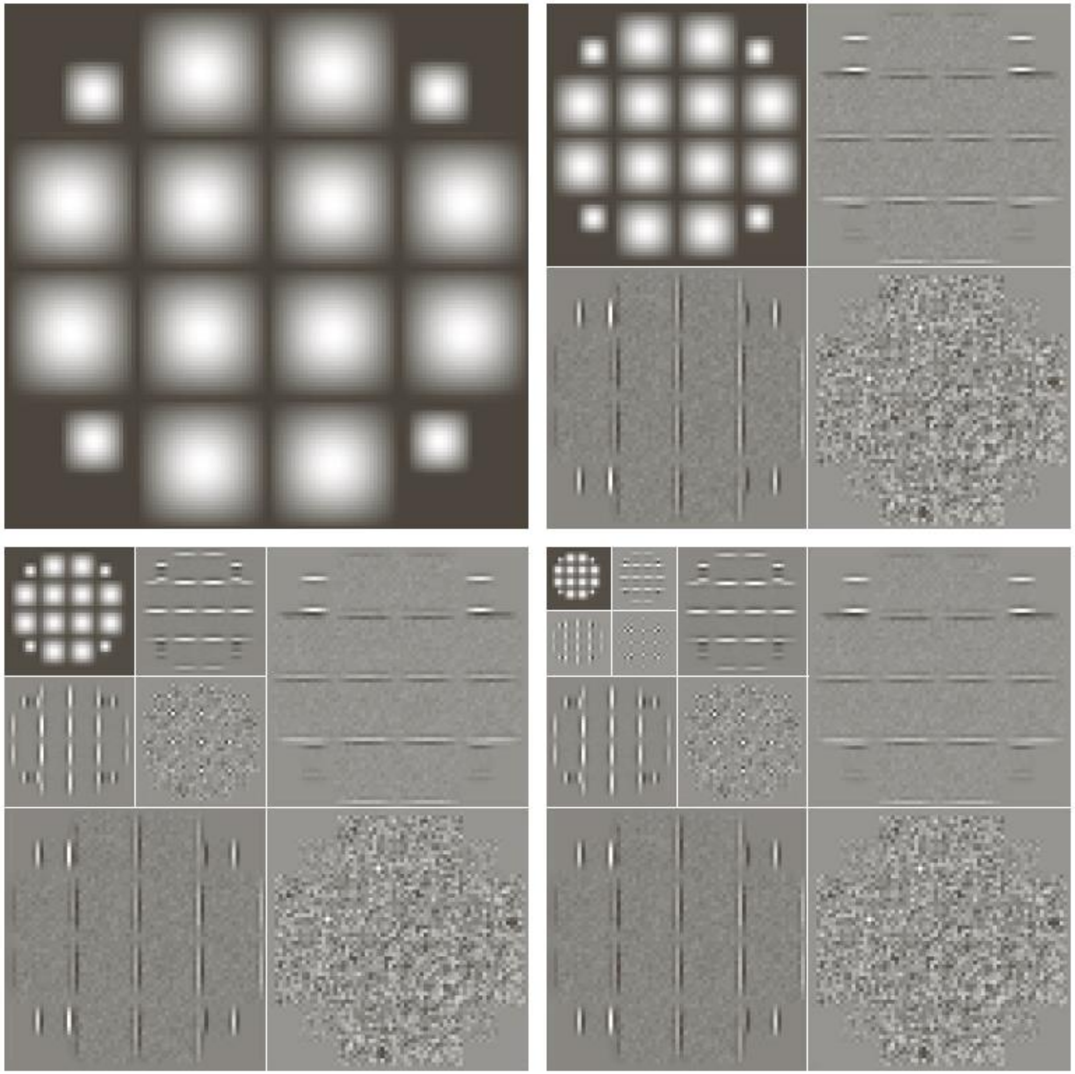


Figure 3.22: 2D wavelet decomposition; another view of the same data as Figure 3.21. © *Cengage Learning 2015.*

2D Wavelet Transform Example



a b
c d

FIGURE 7.25 Computing a 2-D three-scale FWT: (a) the original image; (b) a one-scale FWT; (c) a two-scale FWT; and (d) a three-scale FWT.